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Eigen polynomial of Square Matrix in Terms of Determinants

Surinder Mohan Deep

Associate Prof. in Mathematics, Govind National College, Narangwal

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ABSTRACT

This paper is an attempt to find the characteristic equation of a square matrix in terms of its determinants. I have proved this result upto n = 4, rest similar procedure can be adopted.

Square matrix: A rectangular matrix $[a_{ij}]_{mxn}$ of order m x n is said to be square matrix if m = n
Determinant: Determinant of a square matrix is a number associated to it.
Characteristic equation: If $A=[a_{ij}]_{nxn}$ is square matrix of order n then equation $|A-\lambda I| = 0$ is its characteristic equation

If $A=[a_{ij}]_{nxn}$ is square matrix of order n then its characteristic equation is given by $|A-\lambda I| = 0$
i.e. $(-1)^n\lambda^n + (-1)^{n-1}\sum_{i=1}^{n-1} |a_{ii}|\lambda^{n-1} + (-1)^{n-2}\sum_{i=1}^{n-1} \sum_{j=2}^{n} (i < j)\left|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}\right|\lambda^{n-2} + (-1)^{n-3}\sum_{i=1}^{n-2} \sum_{j=2}^{n-1} \sum_{k=3}^{n} (i < j < k)\left|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}\right|\lambda^{n-3} + (-1)^{n-4}\lambda^{n-4}\sum_{i=1}^{n-3} \sum_{j=2}^{n-2} \sum_{k=3}^{n-1} \sum_{l=3}^{n} (i < j < k < l)\left|a_{ii}\ a_{ij}\ a_{ik}\ a_{il}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{jl}\ a_{ki}\ a_{kj}\ a_{kk}\ a_{kl}\ a_{li}\ a_{lj}\ a_{lk}\ a_{ll}\right| + \ldots + (-1)^0\lambda^0|A| = 0$

Proof :- <u>For n=1</u> $A = [a_{11}]_{1x1}$
Characteristic Equation of matrix A is $|A-\lambda I| = 0$
$|a_{11}-\lambda| = 0$
$a_{11}-\lambda = 0$ i.e $-\lambda + a_{11}= 0$
$(-1)^1\lambda^1+ (-1)^0\lambda^0|A| = 0$
Result is true for n=1

<u>For n=2</u> $A = [a_{ij}]_{2x2} = |a_{11}\ a_{12}\ a_{21}\ a_{22}|$

Characteristic Equation of matrix A is $|A-\lambda I| = 0$
$|[a_{11}\ a_{12}\ a_{21}\ a_{22}] - \lambda[1\ 0\ 0\ 1]| = 0$
$[a_{11} - \lambda\ a_{12}\ a_{21}\ a_{22} - \lambda] = 0$
$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$
$a_{11}a_{22} - \lambda(a_{11} + a_{22}) + \lambda^2 - a_{12}a_{21} = 0$
$\lambda^2 - \lambda(a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) = 0$
$(-1)^2\lambda^2 + (-1)^1\lambda^1[|a_{11}| + |a_{12}|] + (-1)^0\lambda^0|a_{11}\ a_{12}\ a_{21}\ a_{22}| = 0$
$(-1)^2\lambda^2 + (-1)^1\lambda^1\sum_{i=1}^{2} |a_{ii}| + (-1)^0\lambda^0|A| = 0$

<u>For n=3</u> $A = [a_{ij}]_{3x3} = |a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23}\ a_{31}\ a_{32}\ a_{33}|$
Characteristic Equation of matrix is given by $|A-\lambda I| = 0$

$|a_{11} - \lambda\ a_{12}\ a_{13}\ a_{21}\ a_{22} - \lambda\ a_{23}\ a_{31}\ a_{32}\ a_{33} - \lambda| = 0$

Or $(a_{11}-\lambda)\,|a_{22}-\lambda\ a_{23}\ a_{32}\ a_{33}-\lambda\,| - a_{12}\,|a_{21}\ a_{23}\ a_{31}\ a_{33}-\lambda\,| + a_{13}\,|a_{21}\ a_{22}-\lambda\ a_{31}\ a_{32}\,| = 0$

$(a_{11}-\lambda)[(a_{22}-\lambda)(a_{33}-\lambda)-a_{32}a_{23}] - a_{12}(a_{21}a_{33}-a_{21}\lambda-a_{31}a_{23}) + a_{13}(a_{21}a_{32}-a_{22}a_{31}+\lambda a_{31} = 0$

$(a_{11}-\lambda)\,[\lambda^2$

$-\lambda\,(a_{22}+a_{33}) + a_{22}a_{33}-a_{32}a_{23}] - a_{12}(-a_{21}\lambda + a_{21}a_{33}-a_{31}a_{23}) + a_{13}(-a_{22}a_{31}+\lambda a_{31}+a_{21}a_{32}) = 0$

$-\lambda^3+\lambda^2(a_{11}+a_{22}+a_{33})\ -\lambda($

$a_{11}a_{22}+a_{11}a_{33}-a_{32}a_{23}+a_{22}a_{33}) + a_{11}a_{22}a_{33} - a_{11}a_{22}a_{33} + a_{12}a_{21}\lambda + (a_{31}a_{23}a_{12} - a_{12}a_{21}a_{33}) +$

$a_{13}a_{31}\lambda + a_{21}a_{13}a_{32} - a_{13}a_{22}a_{31} + a_{22}a_{33}$

$-\lambda+(a_{11}+a_{22}+a_{33})\lambda^2-\lambda[(a_{11}a_{22}-a_{12}a_{21})+(a_{11}a_{33}-a_{13}a_{31})+(a_{22}a_{33}-a_{23}a_{32})]$

$+(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}+a_{12}a_{23}a_{31}-a_{12}a_{21}a_{33}+a_{21}a_{13}a_{32}-a_{13}a_{22}a_{31}) = 0$

$(-1)^3\lambda^3+(-1)^2\sum_{i=1}^{3}\ |a_{ii}|\lambda^2+(-1)^1\lambda\,|a_{11}\ a_{12}\ a_{21}\ a_{22}\,|+|a_{11}\ a_{13}\ a_{31}\ a_{33}\,|+|a_{22}\ a_{23}\ a_{32}\ a_{33}\,|+(-1)^0\lambda^0$

$|a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23}\ a_{31}\ a_{32}\ a_{33}\,|=0$

$(-1)^3\lambda^3+(-1)^2\sum_{i=1}^{3}\ |a_{ii}|\lambda^2+(-1)^1\lambda^1\sum_{i=1}^{2}\ \sum_{j=2}^{3}\ (i<j)\,|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}\,|+(-1)^0\lambda^0|A| = 0$

<u>For n = 4</u> $A = [a_{ij}]_{4x4} = |a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{41}\ a_{42}\ a_{43}\ a_{44}\,|$

Characteristic Equation of A is $|A-\lambda I| = 0$

$|a_{11}-\lambda\ a_{12}\ a_{13}\ a_{14}\ a_{21}\ a_{22}-\lambda\ a_{23}\ a_{24}\ a_{31}\ a_{32}\ a_{33}-\lambda\ a_{34}\ a_{41}\ a_{42}\ a_{43}\ a_{44}-\lambda\,|=0$

$-a_{41}|a_{12}\ a_{13}\ a_{14}\ a_{22}-\lambda\ a_{23}\ a_{24}\ a_{32}\ a_{33}-\lambda\ a_{34}\,| + a_{42}|a_{11}-\lambda\ a_{13}\ a_{14}\ a_{21}\ a_{23}\ a_{24}\ a_{31}\ a_{33}-\lambda\ a_{34}\,| - a_{43}|a_{11}-\lambda\ a_{12}\ a_{14}\ a_{21}\ a_{22}-\lambda\ a_{24}\ a_{31}\ a_{32}\ a_{34}\,| + (a_{44}-\lambda)|a_{11}-\lambda\ a_{12}\ a_{13}\ a_{21}\ a_{22}-\lambda\ a_{23}\ a_{31}\ a_{32}\ a_{33}-\lambda\,|$

$=0$

$-a_{41}[a_{14}\lambda^2+(a_{12}a_{24}+a_{13}a_{34}-a_{14}a_{22}-$

$a_{14}a_{33})\,\lambda+(a_{12}a_{23}a_{34}-a_{12}a_{24}a_{33}+a_{13}a_{32}a_{24}-a_{13}a_{22}a_{34}+a_{14}a_{22}a_{33}-a_{14}a_{23}a_{32})+a_{42}[-a_{24}\lambda^2+($

$a_{33}a_{24}-a_{23}a_{34}+a_{11}a_{24}-$

$a_{14}a_{21})\,\lambda+(a_{11}a_{23}a_{34}-a_{11}a_{33}a_{24}-a_{13}a_{21}a_{34}+a_{13}a_{31}a_{24}+a_{14}a_{21}a_{33}-a_{14}a_{31}a_{23})]-a_{43}[a_{34}\lambda^2+($

$a_{32}a_{24}-a_{22}a_{34}-a_{11}a_{34}+$

$a_{14}a_{31})\,\lambda\ +(a_{11}a_{22}a_{34}-a_{11}a_{32}a_{24}+a_{21}a_{31}a_{24}-a_{12}a_{21}a_{34}+a_{14}a_{21}a_{32}-a_{14}a_{31}a_{22})]$

$+(a_{44}-\lambda)\,[-\lambda^3+(a_{11}+a_{22}+a_{33})\,\lambda^2-(a_{11}a_{22}-a_{12}a_{21}+a_{11}a_{33}-a_{13}a_{31}+a_{22}a_{33}-a_{23}a_{32})$

$+(a_{11}a_{22}a_{33}-a_{11}a_{23}a_{32}+a_{12}a_{23}a_{31}-a_{12}a_{21}a_{33}+a_{21}a_{13}a_{32}-a_{13}a_{22}a_{31})]=0$

$\lambda^4-(a_{11}+a_{22}+a_{33}+a_{44})\,\lambda^3+\lambda^2[(a_{11}a_{22}-a_{12}a_{21})+(a_{13}a_{33}-a_{13}a_{31})+(a_{11}a_{44}-a_{14}a_{41})+$

$(a_{22}a_{33}-a_{23}a_{32})+(a_{22}a_{44}-a_{24}a_{42})+(a_{33}a_{44}-a_{34}a_{43})]$

$-\lambda[(a_{11}a_{22}a_{33}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{31}a_{22}a_{13}-a_{32}a_{23}a_{11}-a_{12}a_{21}a_{33})$

$+(a_{11}a_{22}a_{44}+a_{12}a_{24}a_{41}+a_{42}a_{14}a_{21}-a_{41}a_{14}a_{22}-a_{42}a_{24}a_{11}-a_{21}a_{12}a_{44})$

$+(a_{11}a_{33}a_{44}+a_{13}a_{34}a_{43}+a_{43}a_{14}a_{31}-a_{14}a_{33}a_{41}-a_{43}a_{34}a_{11}-a_{13}a_{31}a_{44})$

$+(a_{22}a_{33}a_{44}+a_{42}a_{23}a_{34}+a_{32}a_{24}a_{43}-a_{42}a_{24}a_{33}-a_{34}a_{43}a_{22}-a_{23}a_{32}a_{44})]$

$+[(a_{11}a_{22}a_{33}a_{44}+a_{12}a_{23}a_{31}a_{44}+a_{21}a_{13}a_{32}a_{44}-a_{11}a_{23}a_{32}a_{44}-a_{12}a_{21}a_{33}a_{44}-a_{13}a_{22}a_{31}a_{44})$

$+(a_{11}a_{23}a_{34}a_{42}+a_{14}a_{42}a_{21}a_{33}+a_{13}a_{31}a_{24}a_{42}-a_{11}a_{33}a_{24}a_{42}-a_{13}a_{21}a_{34}a_{42}-a_{14}a_{31}a_{23}a_{42})$

$+(a_{12}a_{24}a_{33}a_{41}+a_{13}a_{22}a_{34}a_{41}+a_{14}a_{41}a_{32}a_{23}-a_{12}a_{23}a_{34}a_{41}-a_{41}a_{13}a_{32}a_{24}-a_{14}a_{22}a_{33}a_{41})$

$+(a_{11}a_{32}a_{24}a_{43}+a_{12}a_{21}a_{34}a_{43}+a_{14}a_{31}a_{22}a_{43}-a_{11}a_{22}a_{34}a_{43}-a_{12}a_{31}a_{24}a_{43}-a_{43}a_{14}a_{21}a_{32})]=0$

$(-1)^4\lambda^4+(-1)^3\lambda^3\,(|a_{11}|+|a_{22}|+|a_{33}|+|a_{44}|)$

$+(-1)^2\lambda^2\,|a_{11}\ a_{12}\ a_{21}\ a_{22}\,|+|a_{11}\ a_{13}\ a_{31}\ a_{33}\,|+|a_{11}\ a_{14}\ a_{41}\ a_{44}\,|+|a_{22}\ a_{23}\ a_{32}\ a_{33}\,|$

$+|a_{22}\ a_{24}\ a_{42}\ a_{44}\,|+|a_{33}\ a_{34}\ a_{43}\ a_{44}\,|$

+(-

$1)^1\lambda^1$

$|a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23}\ a_{31}\ a_{32}\ a_{33}| + |a_{11}\ a_{13}\ a_{14}\ a_{31}\ a_{33}\ a_{34}\ a_{41}\ a_{43}\ a_{44}| +$
$|a_{11}\ a_{12}\ a_{14}\ a_{21}\ a_{22}\ a_{24}\ a_{41}\ a_{42}\ a_{44}| + |a_{22}\ a_{23}\ a_{24}\ a_{32}\ a_{33}\ a_{34}\ a_{42}\ a_{43}\ a_{44}|$
$+(-1)^0\lambda^0|A| = 0$

$(-1)^4\lambda^4+(-1)^3\lambda^3 \sum_{i=1}^{4} |a_{ii}|+(-1)^2\lambda^2 \sum_{i=1}^{3} \sum_{j=2}^{4} (i<j)|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}|+(-1)^1\lambda^1$
$\sum_{i=1}^{2} \sum_{j=2}^{3} \sum_{k=3}^{4} (i<j<k)|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}|+(-1)^0\lambda^0|A| = 0$
and so on.

Generalizing the result
We get, characteristic equation of matrix A is
$(-1)^n\lambda^n+(-1)^{n-1}\lambda^{n-1} \sum_{i=1}^{n} |a_{ii}|+(-1)^{n-2}\lambda^{n-2} \sum_{i=1}^{n-1} \sum_{j=2}^{n} (i<j)|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}|+(-1)^{n-3}\lambda^{n-3}$
$\sum_{i=1}^{n-2} \sum_{j=2}^{n-1} \sum_{k=3}^{n} (i<j<k)|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}|$
$+(-1)^{n-4}\lambda^{n-4}$
$\sum_{i=1}^{n-3} \sum_{j=2}^{n-2} \sum_{k=3}^{n-1} \sum_{l=3}^{n} (i<j<k<l)|a_{ii}\ a_{ij}\ a_{ik}\ a_{il}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{jl}\ a_{ki}\ a_{kj}\ a_{kk}\ a_{kl}\ a_{li}\ a_{lj}\ a_{lk}\ a_{ll}| + \ldots +(-1)^0\lambda^0|A| = 0$

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