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Eigen polynomial of 4*4 and 5*5 matrices in terms of determinants of their submatrices

Surinder Mohan Deep

Associate Professor in Mathematics, G.N.College, Narangwal

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ABSTRACT

The method to find the characteristic equation of a square matrix of any order is to solve the equation det.$(A - \lambda I) = 0$ i.e. $|A - \lambda I| = 0$ which takes so much time when n = 4,5,…..& so on. This paper is an attempt to simplify the method to find the characteristic equation of a square matrix A of order 4 and 5 by taking the determinants of its all submatrices starting from 1x1 upto nxn.

Definitions:

Square matrix: A rectangular matrix $[a_{ij}]_{mxn}$ of order m x n is said to be square matrix if m = n
Determinant: Determinant of a square matrix is a number associated to it.
Characteristic equation: If $A=[a_{ij}]_{nxn}$ is square matrix of order n then equation $|A-\lambda I| = 0$ is its characteristic equation

Result:

If $A=[a_{ij}]_{nxn}$ is square matrix of order n then its characteristic equation is given by $|A-\lambda I| = 0$

i.e. $(-1)^n\lambda^n + (-1)^{n-1}\sum_{i=1}^{n} |a_{ii}|\lambda^{n-1} + (-1)^{n-2}\sum_{i=1}^{n-1}\sum_{j=2}^{n} (i<j)|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}|\lambda^{n-2}$
$+ (-1)^{n-3}\sum_{i=1}^{n-2}\sum_{j=2}^{n-1}\sum_{k=3}^{n} (i<j<k)|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}|\lambda^{n-3}$
$+ (-1)^{n-4}\lambda^{n-4}\sum_{i=1}^{n-3}\sum_{j=2}^{n-2}\sum_{k=3}^{n-1}\sum_{l=4}^{n} (i<j<k<l)|a_{ii}\ a_{ij}\ a_{ik}\ a_{il}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{jl}\ a_{ki}\ a_{kj}\ a_{kk}\ a_{kl}\ a_{li}\ a_{lj}\ a_{lk}\ a_{ll}| + \ldots + (-1)^0\lambda^0|A| = 0$

Corllary 1: For $A = [a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{41}\ a_{42}\ a_{43}\ a_{44}]$,
the characteristic equation is
$(-1)^4\lambda^4 + (-1)^{4-1}\sum_{i=1}^{4} |a_{ii}|\lambda^{4-1} + (-1)^{4-2}\sum_{i=1}^{4-1}\sum_{j=2}^{4} (i<j)|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}|\lambda^{4-2} + (-1)^{4-3}\sum_{i=1}^{4-2}\sum_{j=2}^{4-1}\sum_{k=3}^{4} (i<j<k)|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}|\lambda^{4-3} + (-1)^0\lambda^0|A| = 0$

Or $(-1)^4\lambda^4 + (-1)^3(a_{11}+a_{22}+a_{33}+a_{44})\lambda^3$
$+(-1)^2(|a_{11}\ a_{12}\ a_{21}\ a_{22}| + |a_{11}\ a_{13}\ a_{31}\ a_{33}| + |a_{11}\ a_{14}\ a_{41}\ a_{44}| + |a_{22}\ a_{23}\ a_{32}\ a_{33}| + |a_{22}\ a_{24}\ a_{42}\ a_{44}| + |a_{33}\ a_{34}\ a_{43}\ a_{44}|)\lambda^2$
$+ (-1)^1$
$(|a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23}\ a_{31}\ a_{32}\ a_{33}| + |a_{11}\ a_{12}\ a_{14}\ a_{21}\ a_{22}\ a_{24}\ a_{41}\ a_{42}\ a_{44}| + |a_{11}\ a_{13}\ a_{14}\ a_{31}\ a_{33}\ a_{34}\ a_{41}\ a_{43}\ a_{44}| +$
$|a_{22}\ a_{23}\ a_{24}\ a_{32}\ a_{33}\ a_{34}\ a_{42}\ a_{43}\ a_{44}|)\lambda^1$
$+ (-1)^0|a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{41}\ a_{42}\ a_{43}\ a_{44}|\lambda^0 = 0$

Example: Take $A = [3\ -3\ 5\ 1\ 1\ -2\ 3\ 0\ 4\ 5\ 2\ 9\ -2\ -5\ -1\ 1]$
Then characteristic equation of A is given by $|A - \lambda I| = 0$
or $|3-\lambda\ \ -3\ 5\ 1\ 1\ \ -2-\lambda\ 3\ 0\ 4\ 5\ 2-\lambda\ 9\ -2\ -5\ -1\ 1-\lambda| = 0$
or $(3-\lambda)|-2-\lambda\ 3\ 0\ 5\ 2-\lambda\ 9\ -5\ -1\ 1-\lambda| + 3|1\ 3\ 0\ 4\ 2-\lambda\ 9\ -2\ -1\ 1-\lambda| + 5|1\ -2-\lambda\ 0\ 4\ 5\ 9\ -2\ -5\ 1-\lambda| - 1|1\ -2-\lambda\ 3\ 4\ 5\ 2-\lambda\ -2\ -5\ -1| = 0$
or $(3-\lambda)(-\lambda^3 + \lambda^2 + 10\lambda - 172) + 3(\lambda^2 + 9\lambda - 55) + 5(-4\lambda^2 + 9\lambda + 94) - 1(-2\lambda^2 - 9\lambda - 25) = 0$
or $\lambda^4 - 4\lambda^3 - 22\lambda^2 + 283\lambda - 186 = 0$

Using the above result
$(-1)^4\lambda^4 + (-1)^3(3+(-2)+2+1)\lambda^3$
$+(-1)^2(|3\ -3\ 1\ -2| + |3\ 5\ 4\ 2| + |3\ 1\ -2\ 1| + |-2\ 3\ 5\ 2| + |-2\ 0\ -5\ 1| + |2\ 9\ -1\ 1|)\lambda^2$

$+ (-1)^1$ ($|3\ -3\ 5\ 1\ -2\ 3\ 4\ 5\ 2\ |$+$|3\ -3\ 1\ 1\ -2\ 0\ -2\ -5\ 1\ |$+$|3\ 5\ 1\ 4\ 2\ 9\ -2\ -1\ 1\ |$+$|-2\ 3\ 0\ 5\ 2\ 9\ -5\ -1\ 1\ |$)$\lambda^1$
+ $|3\ -3\ 5\ 1\ 1\ -2\ 3\ 0\ 4\ 5\ 2\ 9\ -2\ -5\ -1\ 1\ |$= 0
We get the same result $\lambda^4 - 4\lambda^3 - 22\lambda^2 + 283\lambda - 186 = 0$

Corllary 2: For A = $[a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45}\ a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55}\ ]$,
the characteristic equation is
$(-1)^5\lambda^5 + (-1)^{5-1}\sum_{i=1}^{5}\ |a_{ii}|\lambda^{5-1} + (-1)^{5-2}\sum_{i=1}^{5-1}\ \sum_{j=2}^{5}\ (i<j)\left|a_{ii}\ a_{ij}\ a_{ji}\ a_{jj}\ \right|\lambda^{5-2}$
$+ (-1)^{5-3}\sum_{i=1}^{5-2}\ \sum_{j=2}^{5-1}\ \sum_{k=3}^{5}\ (i<j<k)\left|a_{ii}\ a_{ij}\ a_{ik}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{ki}\ a_{kj}\ a_{kk}\ \right|\lambda^{5-3}$
$+(-1)^{5-4}\sum_{i=1}^{5-3}\ \sum_{j=2}^{5-2}\ \sum_{k=3}^{5-1}\ \sum_{l=4}^{5}\ (i<j<k<l)\left|a_{ii}\ a_{ij}\ a_{ik}\ a_{il}\ a_{ji}\ a_{jj}\ a_{jk}\ a_{jl}\ a_{ki}\ a_{kj}\ a_{kk}\ a_{kl}\ a_{li}\ a_{lj}\ a_{lk}\ a_{ll}\ \right|\lambda^{5-4}+(-1)^{5-5}\lambda^{5-5}|A| = 0$

Or $(-1)^5\lambda^5 + (-1)^4(a_{11}+a_{22}+a_{33}+a_{44}+a_{55})\lambda^4$
$+(-1)^3(|a_{11}\ a_{12}\ a_{21}\ a_{22}\ |+|a_{11}\ a_{13}\ a_{31}\ a_{33}\ |+|a_{11}\ a_{14}\ a_{41}\ a_{44}\ |+|a_{11}\ a_{15}\ a_{51}\ a_{55}\ |+|a_{22}\ a_{23}\ a_{32}\ a_{33}\ |+|a_{22}\ a_{24}\ a_{42}\ a_{44}\ |+$
$|a_{22}\ a_{25}\ a_{52}\ a_{55}\ |+|a_{33}\ a_{34}\ a_{43}\ a_{44}\ |+|a_{33}\ a_{35}\ a_{53}\ a_{55}\ |+|a_{44}\ a_{45}\ a_{54}\ a_{55}\ |)\lambda^3$
$+ (-1)^2$
$(|a_{11}\ a_{12}\ a_{13}\ a_{21}\ a_{22}\ a_{23}\ a_{31}\ a_{32}\ a_{33}\ |+|a_{11}\ a_{12}\ a_{14}\ a_{21}\ a_{22}\ a_{24}\ a_{41}\ a_{42}\ a_{44}\ |+|a_{11}\ a_{12}\ a_{15}\ a_{21}\ a_{22}\ a_{25}\ a_{51}\ a_{52}\ a_{55}\ |+$
$|a_{11}\ a_{13}\ a_{14}\ a_{31}\ a_{33}\ a_{34}\ a_{41}\ a_{43}\ a_{44}\ |+|a_{11}\ a_{13}\ a_{15}\ a_{31}\ a_{33}\ a_{35}\ a_{51}\ a_{53}\ a_{55}\ |$
$+|a_{11}\ a_{14}\ a_{15}\ a_{41}\ a_{44}\ a_{45}\ a_{51}\ a_{54}\ a_{55}\ |+|a_{22}\ a_{23}\ a_{24}\ a_{32}\ a_{33}\ a_{34}\ a_{42}\ a_{43}\ a_{44}\ |+|a_{22}\ a_{23}\ a_{25}\ a_{32}\ a_{33}\ a_{35}\ a_{52}\ a_{53}\ a_{55}\ |+$
$|a_{22}\ a_{24}\ a_{25}\ a_{42}\ a_{44}\ a_{45}\ a_{52}\ a_{54}\ a_{55}\ |+|a_{33}\ a_{34}\ a_{35}\ a_{43}\ a_{44}\ a_{45}\ a_{53}\ a_{54}\ a_{55}\ |)\lambda^2$
$+ (-1)^1(|a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{41}\ a_{42}\ a_{43}\ a_{44}\ |+$
$|a_{11}\ a_{12}\ a_{13}\ a_{15}\ a_{21}\ a_{22}\ a_{23}\ a_{25}\ a_{31}\ a_{32}\ a_{33}\ a_{35}\ a_{51}\ a_{52}\ a_{53}\ a_{55}\ |+$
$|a_{11}\ a_{12}\ a_{14}\ a_{15}\ a_{21}\ a_{22}\ a_{24}\ a_{25}\ a_{41}\ a_{42}\ a_{44}\ a_{45}\ a_{51}\ a_{52}\ a_{54}\ a_{55}\ |$
$+|a_{11}\ a_{13}\ a_{14}\ a_{15}\ a_{31}\ a_{33}\ a_{34}\ a_{35}\ a_{41}\ a_{43}\ a_{44}\ a_{45}\ a_{51}\ a_{53}\ a_{54}\ a_{55}\ |+$
$|a_{22}\ a_{23}\ a_{24}\ a_{25}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ a_{42}\ a_{43}\ a_{44}\ a_{45}\ a_{52}\ a_{53}\ a_{54}\ a_{55}\ |)\lambda^1+(-1)^0|a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15}\ a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25}\ a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35}\ a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45}\ a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55}\ |\lambda^0 = 0$

Example: Take A = $[-2\ 3\ 5\ 6\ -1\ 9\ 7\ -1\ 0\ 3\ -4\ -2\ 1\ 3\ 2\ 5\ 9\ 3\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ ]$
Then characteristic equation of A is given by $|A - \lambda I| = 0$
Or $|-2-\lambda\ 3\ 5\ 6\ -1\ 9\ 7-\lambda\ -1\ 0\ 3\ -4\ -2\ 1-\lambda\ 3\ 2\ 5\ 9\ 3\ 2-\lambda\ 3\ 4\ 5\ 6\ 1\ 2-\lambda\ |$ = 0
Or $(-2-\lambda)\ |7-\lambda\ -1\ 0\ 3\ -2\ 1-\lambda\ 3\ 2\ 9\ 3\ 2-\lambda\ 3\ 5\ 6\ 1\ 2-\lambda\ |-3|9\ -1\ 0\ 3\ -4\ 1-\lambda\ 3\ 2\ 5\ 3\ 2-\lambda\ 3\ 4\ 6\ 1\ 2-\lambda\ |$
$+5|9\ 7-\lambda\ 0\ 3\ -4\ -2\ 3\ 2\ 5\ 9\ 2-\lambda\ 3\ 4\ 5\ 1\ 2-\lambda\ |-6|9\ 7-\lambda\ -1\ 3\ -4\ -2\ 1-\lambda\ 2\ 5\ 9\ 3\ 3\ 4\ 5\ 6\ 2-\lambda\ |$
$+(-1)\ |9\ 7-\lambda\ -1\ 0\ -4\ -2\ 1-\lambda\ 3\ 5\ 9\ 3\ 2-\lambda\ 4\ 5\ 6\ 1\ |$ = 0
Or $(-2-\lambda)(\lambda^4-12\lambda^3+2\lambda^2+192\lambda^1-288)-3(-9\lambda^3+29\lambda^2+276\lambda^1-112)+5(-4\lambda^3+49\lambda^2+164\lambda^1+16)-6(-5\lambda^3-31\lambda^2+84\lambda^1+720)-1(4\lambda^3-14\lambda^2+160\lambda^1+608) = 0$
Or $-\lambda^5 + 10\lambda^4+55\lambda^3+162\lambda^2-768\lambda^1-3936 = 0$

Use the above result
$(-1)^5\lambda^5 + (-1)^4(-2+7+1+2+2)\lambda^4$
$+(-1)^3(|-2\ 3\ 9\ 7\ |+|-2\ 5\ -4\ 1\ |+|-2\ 6\ 5\ 2\ |+|-2\ -1\ 4\ 2\ |+|7\ -1\ -2\ 1\ |+|7\ 0\ 9\ 2\ |+$
$|7\ 3\ 5\ 2\ |+|1\ 3\ 3\ 2\ |+|1\ 2\ 6\ 2\ |+|2\ 3\ 1\ 2\ |)\lambda^3$
$+ (-1)^2$
$(|-2\ 3\ 5\ 9\ 7\ -1\ -4\ -2\ 1\ |+|-2\ 3\ 6\ 9\ 7\ 0\ 5\ 9\ 2\ |+|-2\ 3\ -1\ 9\ 7\ 3\ 4\ 5\ 2\ |+|-2\ 5\ 6\ -4\ 1\ 3\ 5\ 3\ 2\ |+|-2\ 5\ -1\ -4\ 1\ 2\ 4\ 6\ 2\ |$
$+|-2\ 6\ -1\ 5\ 2\ 3\ 4\ 1\ 2\ |+|7\ -1\ 0\ -2\ 1\ 3\ 9\ 3\ 2\ |+|7\ -1\ 3\ -2\ 1\ 2\ 5\ 6\ 2\ |+|7\ 0\ 3\ 9\ 2\ 3\ 5\ 1\ 2\ |+|1\ 3\ 2\ 3\ 2\ 3\ 6\ 1\ 2\ |)\lambda^2$
$+ (-1)^1(|-2\ 3\ 5\ 6\ 9\ 7\ -1\ 0\ -4\ -2\ 1\ 3\ 5\ 9\ 3\ 2\ |+|-2\ 3\ 5\ -1\ 9\ 7\ -1\ 3\ -4\ -2\ 1\ 2\ 4\ 5\ 6\ 2\ |+$
$|-2\ 3\ 6\ -1\ 9\ 7\ 0\ 3\ 5\ 9\ 2\ 3\ 4\ 5\ 1\ 2\ |$ $+|-2\ 5\ 6\ -1\ -4\ 1\ 3\ 2\ 5\ 3\ 2\ 3\ 4\ 6\ 1\ 2\ |+|7\ -1\ 0\ 3\ -2\ 1\ 3\ 2\ 9\ 3\ 2\ 3\ 5\ 6\ 1\ 2\ |)\lambda^1+(-1)^0|-2\ 3\ 5\ 6\ -1\ 9\ 7\ -1\ 0\ 3\ -4\ -2\ 1\ 3\ 2\ 5\ 9\ 3\ 2\ 3\ 4\ 5\ 6\ 1\ 2\ |\lambda^0 = 0$
Or $(-1)^5\lambda^5 + (-1)^4(10)\lambda^4+(-1)^3(-55)\lambda^3+(-1)^2(162)\lambda^2+(-1)^1(768)\lambda^1+(-1)^0(-3936)\lambda^0 = 0$
Or $-\lambda^5 + 10\lambda^4+55\lambda^3+162\lambda^2-768\lambda^1-3936 = 0$

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