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Optimal Plans using Projective Geometry

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ABSTRACT

The paper discusses the construction of two new specific family of optimal fractional factorial plans which permit the estimability of mean, all main effects (ME) and a specified set of two factor interactions (2FI) using cap and finite projective geometry. Dey and Suen (2002) obtained several families of optimal plans using projective geometry, modifying the theorem of Dey and Mukerjee (1999b) to construct universally optimal plans under hierarchical model for estimation of the mean, all main effect and a specified set of two factor interaction.

1. Introduction: The optimality of fractional factorial plans and it applications has been studied by many researchers (Aggarwal et al 2008, Aggarwal et al 2005, Dey et al 2002, Dey et al 1999b). We construct two new specific family of optimal fractional factorial plans in Section 3 and Section 4 using concept of largest binary cap and concept of flat in projective geometry respectively. In Section 2 we present some essential preliminaries of finite projective geometry relevant to this paper.

2. Finite Projective Geometry

In Hirschfeld (1998)," PG (r-1, m) is defines as (r-1) dimensional finite projective geometry over GF(m) , Galois field of order m , m is a prime power, consists of the ordered non null set $(x_0,x_1,.....x_{r-1})$ of points where x_i (i= 0,1...r-1) are elements of G F(m). For any non-zero $\lambda \in$ GF(m) the point $(\lambda x_0$,......... $\lambda x_{r-1})$ represents the same point as that of $(x_0,..............x_{r-1})$. All those points which satisfy a set of (r-t-1) linearly independent homogeneous equations with coefficients from GF(m) (all of them are not simultaneously zero within the same equation) is said to represent a t-flat in PG (r-1, m)). The number of points lying on a (t – 1)-flat in PG$(r-1, m)$ is $\frac{(m^t-1)}{(m-1)}$. The number of (r-2) flats within a (r-1) dimensional finite projective geometry over GF(m) which contains a given (r-3) flat is (m+1)".

Let cardinality of binary cap is n. Let G denote the set of n-points in r-1 dimensional projective geometry over GF (2) satisfies the following conditions:
(i) the set G consists of n-distinct points and is not contained in a proper subspace;
(ii) two points of G contain in a line but no three points contain in a line;
(iii) there is a plane containing four points of the set G.
Then P is a matrix of order r x n formed by the points of G. This is also established that the largest size cap in PG(r-1,2) is 2^{r-1}.

3. Construction using caps

Theorem 3.1. Let D be the class of N-run fractional factorial plans for an arbitrary factorial plans for an arbitrary factorial experiment involving n-factors such that each member of D allows the estimability of the mean, the main effects $F_1,F_2;F_3,F_4;.........F_n$ and k=two factor interactions $F_{i_1} F_{j_1}$,....... $F_{i_k} F_{j_k}$ where $1 \le i_b , j_b \le n$.

Dey and Mukerjee (1999b) stated that "under a hierarchical model, a plan $d \in D$ has inter- effect orthogonality and hence universally optimal over D if all level combinations of the following sets of factors appear equally often in d:

a) $\{F_u, F_v\}$, $1 \le u < v \le n$;

(b) $\{F_u, F_{i_b}, F_{j_b}\}$, $1 \le u \le n$, $1 \le b \le k$;

(c) $\{F_{i_b}, F_{j_b}, F_{i_g}, F_{j_g}\}$, $1 \le b < g \le k$,

where a factor is counted only once if it is repeated in (b) and (c)".

Theorem 3.2: Let $F_1,F_2,F_3,.........F_n$ be distinct points in PG (r-1,m) and let P be a r x n matrix with columns $F_1,F_2;F_3;.........F_n$. For every b= 1, ...k, let F_{i_b} , F_{j_b}

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$(1 \le i_b , j_b \le n)$ denote the set of (m-1) other points on the line containing F_{i_b} and F_{j_b}. If the n+ k(m-1) points $F_1,F_2;F_3;.........F_n$, $F_{i_1}F_{j_1}$,....... $F_{i_k}F_{j_k}$ are all distinct, then all linear combination of rows of matrix P over GF(2) satisfies the condition of Theorem 3.1. Dey and Suen (2002) have proved the theorem using the result of Bose and Bush (1952).

Based on Theorem 3.2, we now construct the specific family of optimal plans. We introduce the following notations:

A plan allowing the optimal estimation of mean, u-main effects F_1, $F_2,F_3;.........F_u$ and u-1 two-factor interactions F_1F_2 , F_1F_3 ....$F_1 F_u$ will be denoted by $(F_1 \mid F_2 ....F_u)$.

Theorem 3.3: Let the largest size of binary cap is n. Let P be a matrix of order r x n , where n columns are the points of the cap of the projective geometry of dimension r over GF (2).Then the linear combination of rows of matrix P over GF(2) is the universally optimal plan d for an 2^n experiment involving 2^r runs where $d \equiv \{ F_1 \mid F_2 ....F_n \}$.

Proof: The largest size of binary cap in PG(r-1,2) is 2^{r-1} . Thus, in this case, n= 2^{r-1} . P be a matrix of order r x n formed by the points of the largest binary cap . Since, in the caps, no three points are collinear, thus if we fix any one point in caps and interact the other points of caps with the fixed point, then the n= 2^{r-1} main effects F_i $(1 \le i \le n)$ and the n-1 = 2^{r-1} - 1 two-factor interactions $F_1 F_{i_c}$ $(2 \le c \le n)$ satisfying the condition of Theorem 3.2 . Hence, we obtain an optimal plan d generated by the row space of matrix P for a $2^n = 2^{2^{r-1}}$ experiment involving 2^r runs for estimating mean, all n= 2^{r-1} main effects and n-1 = 2^{r-1} -1 two factor interactions. We explain the method through example**.**

Example 3.1: Consider 2^4 =16 cap is the largest cap in PG (4,2) .16 points in PG (4,2) form a matrix P of order 5 x 16.

$$P = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{vmatrix}$$

The row space of the matrix P generates an optimal plan for estimating mean, the main effects $F_1,F_2;F_3,.........F_{16}$ and 15 two factor interactions F_1F_2 , F_1F_3 ....$F_1 F_{16}$ as the 16 main effects and 15 two factor interactions are distinct points of PG(4,2). In Table 1 we give some optimal fractional factorial plans using the largest binary caps.

Table 1

S. No	PG (r-1, 2)	Largest cap	Optimal plan for u ME and u-1 2FI
1.	PG (2,2)	4	. u=4, u-1=3
2.	PG (3,2)	8	u=8 ,u-1= 7
3.	PG (4,2)	16	u=16 ,u-1= 15
4.	PG (5,2)	32	u=32, u-1=31
5.	PG (6,2)	64	u=64 ,u-1=63
6.	PG (7,2)	128	u=128,u-1= 127

4. Optimal Plans using finite projective geometry

Now using the theorems of Dey and Suen (2002) and the following properties of projective geometry of Hirschfeld (1998), we construct the optimal plan:

Properties 4.1. Consider a (r-1) dimensional finite projective geometry PG (r-1, m) over GF(m). The number of (r-2) flats containing a (r-3) flat is (m+1). Let Q be the set of the points of (r-3) flat. The cardinality of Q is $(m^{r-2}-1)/(m-1)$. S is a set of (m+1) points obtained by choosing any one point from each of the (r-2) flat. The cardinality of S is (m+1). Let R be a set of the union of set Q and set S. Each point of Q interacts with every point of S. $S \cup Q = R$ and cardinality of R is f= $(m^{r-2}-1)/(m-1)$ +(m+1).) **Theorem 4.2:** An universally optimal fractional factorial plan d for an m^f experiment can be obtained by taking linear combination of rows of matrix P of order r x f (the columns of the matrix are points of R) involving m^r runs for estimating f main effects and $[(m^{r-2}-1)/(m-1)][(m+1)]$=g (say) two factor interactions. Illustration with example.

Example 4.1. Consider a 3-dimensional finite projective geometry PG (3,3) over GF (3). Here r = 4, m = 3. Let the points of a 1 flat (line) L be given by (0100, 1000, 1100, 1200). H_1 , H_2 , H_3 and H_4 are the 2 flats containing the 1 flat L. Let the points of the H_1 , H_2 , H_3 and H_4 be given by

H_1: (0010,0100,0110,0120,1000,1010,1020,1100,1110,1120,1200,1210,1220)

H_2: (0001,0100,0101,0102,1000,1001,1002,1100,1101,1102,1200,1201,1202)

H_3: (0012,0100,0112,0121,1000,1012,1021,1100,1112,1121,1200,1212,1221)

H_4:(0011,0100,0111,0122,1000,1011,1022,1100,1111,1122,1200,1211,1222)

Q is a set of points of L. S is a set of 4-points obtained by choosing any one point from each

$H_i \setminus L$ (i = 1,2,3,4). | S | =4, | Q | = 4, | R | = 8

P is a matrix of order 4 x 8 given below:

$$\begin{matrix} F1 & F2 & F3 & F4 & F5 & F6 & F7 & F8 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{matrix}$$

The linear combination of rows of P provide the optimal fractional factorial plan for an 3^8 experiment involving 3^4 runs for estimating mean, 8main effects F_1,………, F_8 ,16 two factor interactions $F_1 F_5$, $F_1 F_6$,……. $F_1 F_8$, ……,F_4F_5 ,……… , F_4F_8.
In Table 2.1, we present some fractional factorial plans.

Table 2 Optimal Fractional Factorial Plans

PG (r-1, m)	Optimal plan for f ME and g 2FI
PG (3 , 2)	f=6 and g=9
PG (4, 2)	f=10 and g= 21
PG (2, 3)	f=5 and g= 4
PG (3, 3)	f=8 and g=16
PG (4, 3)	f=17 and g=52

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