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New look at symmetries and conservation laws

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ABSTRACT

Using the geometric language of modern differential geometry, we discuss different methods for obtaining symmetries and first integrals of Hamiltonian Poisson vector fields which are based on notion of pseudo symmetries, generalized Noether theorem, and Poisson brackets on tangent manifolds. The differential system which describes the two-dimensional isotropic harmonic oscillator is given as example.

1 Introduction

One of the main subjects of dynamics is conservation laws (or first integrals, or constants of motion, or invariant functions) for ordinary differential equations. In this connection,it is well known how to generate conservation laws for differential systems using Noether theorem which associates to every symmetry a conservation law. The aim of this paper is to study the subject of symmetries and conservation laws (for certain definitions and properties see section 2) from several different points of view which appear to be of great interest. One of the methods given in section 3 is based on generalization of the notion of symmetries to adjoint pseudosymmetries and psendosymmetries [1,2,3]. This method has the advantage that in contrast with the Noether theorem its application does not require that the differential system follows from a variational principle, i.e. the critical action principle. Another method given in section 4 generalizes the notion of classical Noetherian symmetries to exact Cartan symmetries on tangent bundles. Finally exact Cartan symmetries are obtained from first integrals of Hamiltonian Poisson vector fields by using notion of Poisson brackets on tangent bundles given in section 5.The last method has the advantage that the existence of Lagrangian or Hamiltonian is not assumed apriori in that case.

2 Poisson Manifolds, Hamiltonian vector fields, and first integrals

In this section we first review the necessary facts on Poisson manifolds. For more details see [4]. Let M be a differentiable manifold of dimension n, $C^\infty(M)$ the ring of real-valued smooth function on M, $\chi(M)$ the Lie algebra of vector fields on M and $\Omega^p(M)$ the space of p - differential froms on M such that $1 \leq p \leq n$.

Definition 2.1 A manifold M together with an internal binary operation on $C^\infty(M)$,called Poisson bracket or structure and denoted by {,} , is called a Poisson manifold if this operation satifies the following four properties :

PB 1 : {, } is real bilinear and antisymmetric

PB 2 : Leibnitz Rule I :

$$\{f, gh\} = \{f, g\}h + g\{f, h\}$$

PB 3 : Jacobi identity :

$$\{f, \{g_1, g_2\}\} = \{\{f, g_1\}, g_2\} + \{g_1, \{f, g_2\}\}$$

By PB 2 and antisymmetry , {,} acts on each factor as a vector field, whence it must be of the form

$$\{f_1, f_2\} = \wedge(df_1, df_2)$$

where $\wedge$ is a skew - symmetric contravariant tensor field of rank 2 ,i.e. a skew- symmetric bilinear form with real values $\wedge(w_1, w_2)$. Conversely, given a skew-symmetric contravariant tensor of rank 2 called Poisson tensor (field) ,we can use the above formula to define a 2- bracket of functions satisfying PB 1 and PB 2. It should be noted that $\wedge$ defines a bundle mapping

$$\sharp \quad : T^*(M) \to T(M)$$

given by

$$< \sharp(\alpha), \beta >= \wedge(\alpha, \beta)$$

for all arguments $\alpha, \beta \in \Omega^1(M)$ .Therefore we are led to the basic notion of the Hamiltonian vector field associated with the Hamiltonian function H defined by :

$$X_H = \sharp(dH)$$

Related to our subject of interest ,namely first integrals, the following information is important.
For $X \in \chi(M)$ with local expression $X = X^i(x)\frac{\partial}{\partial x^i}$ we consider the flow of X which is given by the system of differential equations

$$\frac{dx^i}{dt}(t) = X^i(x^1(t), \dots, x^n(t)) \tag{2.1}$$

and whose solution is called an integral curve of X.

Definition 2.2 A function F $\in C^\infty$ (M) is called a first integral of X or of (2.1) if F is constant by virtue of the solution of (2.1), i.e.

$$\frac{d}{dt}(F \bullet c)(t) = 0$$

for every integral curve c(t) of X .
The following characterization proves very helpful.

Proposition 2.3 $F \in C^\infty(M)$ is a first integral of X or of (2.1) if and only if the Lie derivative of F with respect to X vanishes, i.e.

$$L_X F = 0.$$

For this work, the following three properties are very important.

Property 2.4 A function $F \in C^\infty(M)$ is a first integral of X_H if and only if

$$\{H, F\} = 0$$

Property 2.5 The Poisson bracket of two first integrals of X_H is also a first integral of X_H

Property 2.6 The Hamiltonian vector fields are infinitesimal automorphisms of the Poisson tensor field :

$$L_{X_H} \wedge = 0.$$

3 Adjoint Pseudosymmetry approach to first integrals

In this section we shall describe a method that uses adjoint pseudo-symmetries to find first integrals for vector fields. For this approach we need the following :

Definition 3.1 (i) Let $X \in \chi(M)$. Then a given $Y \in \chi(M)$ is called a symmetry of X if

$$L_X Y = 0,$$

and , a given $Z \in \chi(M)$ is called a Y - pseudo symmetry of X if

$$L_X Z = fY$$

for some $f \in C^\infty(M)$.

(ii) A given $w \in \Omega^1(M)$ is called an adjoint symmetry of X if

$$L_X w = 0$$

and, a given $\alpha \in \Omega^1(M)$ is called an w - adjoint symmetry of X if there exists $\rho \in C^\infty(m)$ such that

$$L_X \alpha = \rho w$$

A straightforward computation leads to the following result on the association between adjoint pseudosymmetries and first integrals.

Proposition 3.2 Let X_H be a Hamiltonian vector field associated with the Hamiltonian function H on a Poisson manifold $(M, \{,\})$ and $w \in \Omega^1(M)$ be an adjoint symmetry of X_H . If $\alpha \in \Omega^1(M)$ is an w - adjoint psuedosymmetry of X_H, then

$$F = \wedge(w, \alpha) \tag{3.1}$$

is a first integral of X_H . In particular , if w, α are adjoint symmetries then the function given by the above formula is first integral of X_H.
Restricting ourselves to exact 1- forms : $w = df, \alpha = df_1$, with $f, f_1 \in C^\infty(M)$, from the definition of $\wedge$ we get that F defined by Proposition 3.2 is

$$F = \wedge(df, df_1) = \{f, f_1\}.$$

Recalling also that $dL_X = L_X d$, we have

Proposition 3.3 Let X_H be a Hamiltonian vector field and $f, f_1 \in C^\infty(M)$ such that

(i) $L_{X_H} f$ is a closed 1-form , i.e. $d(L_{X_H} f) = 0$

(ii) $L_{X_H} df_1 = \rho_1 df$ for some $\rho_1 \in C^\infty(M)$.

Then $F = \{f, f_1\}$
is a first integral for X_H.
Apparently this result represents a generalization of Property 2.5 . This is because if f, f_1 are first integrals of X then (i) and (ii) hold with $\rho_1 = 0$. Thus Property 2.5 meaning that

$$\left.\begin{matrix}\{H, f\} = 0 \\ \{H, f_1\} = 0\end{matrix}\right\} \Rightarrow \{H, \{f, f_1\}\} = 0.$$

is a particular case of Proposition 3.3 meaning that

$$\left.\begin{matrix}d\{H, f\} = 0 \\ d\{H, f_1\} = \rho_1 df\end{matrix}\right\} \Rightarrow \{H, \{f, f_1\}\} = 0.$$

Remark 3.4 Suppose we express $\{f, f_1\}$ in terms of X_f and X_{f_1} by

$$\wedge(df, df_1) = w(X_f, X_{f_1})$$

where w is a covariant skew-symmetric tensor field of rank 2. Given a Hamiltonian vector field X_H , it is now clear from Property 2.5 that w is an invariant 2- form of X_H , i.e. $L_{X_H} w = 0$
We can use this observation to arrive at a result which is similar to Proposition 3.2 .

Proposition 3.5 Let $X_H \in \chi(M)$ be a Hamiltonian vector field and $w(L_1, L_2)$ be the covariant from of $\wedge(w_1, w_2)$.If $Y \in \chi(M)$ is a symmetry of X_H and $S \in \chi(M)$is a Y- pseudosymmetry of X then

$$F = w(Y, S) \tag{3.2}$$

is a first integral of X_H. In particular , if Y, S are symmetries of $X_H$then F given by (3.2) is a first integral.
The results of this section find an interesting applications for a Lagrangian system. In that case we have a natural invariant 2-form, the so called Cartan 2-form.

4 Lagrangian systems, and Noetherian approach to first integral

In this section we shall recall how to go from exact Cartan symmetries to first integrals using the classical Noetherian approach in a Lagrangian framework.
For a Lagrangian system (M,L) let M be an m-dimensional manifold with TM the tangent bundle and L be a smooth function

$$L : TM \to \Re$$

usually called Lagrangian. On TM an important structure is the $C^\infty(TM)$ linear mapping

$$J : \chi(TM) \to \chi(TM)$$

given by

$$J = \frac{\partial}{\partial \dot{x}^i} \otimes dx^i \tag{4.1}$$

where (x^i) are the coordinates on M and $(x^i, \dot{x}^i)$ are the associated coordinates on TM.
To the Lagrangian L we can associated two forms, usually called Cartan forms

$$\theta_L = J^*(dL) \tag{4.2}$$

$$w_L = -d\theta_L \tag{4.3}$$

where J^* is the adjoint structure of J given by (4.1).
It may be recalled that the extremals of the action

$$\int L(x^i, \dot{x}^i, t)dt$$

are solutions of the Eular -Lagrange equations

$$0 = E_i(L) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} \tag{4.4}$$

If in (4.4) we compute the derivative with respect to time we get

$$g_{ij}\ddot{x}^j + \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^k}\dot{x}^k - \frac{\partial L}{\partial x^i} = 0 \tag{4.5}$$

Here w_{ij} called the metric of L is given by

$$g_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j} \tag{4.6}$$

Definition 4.1 The Lagrangian L is called regular or nondegenerate if the metric (g_{ij}) is invertible , that is $det(g_{ij}) \neq 0$
If L is regular we may denote by (g^{ij}) the inverse of (g_{ij}). By multiphication of (4.5) with (g^{ij}) we are led to

Proposition 4.2 If the Lagrangian L is regular then the extremals of the action $\int Ldt$ are solutions of second order equations

$$\ddot{x}^i = f^i \tag{4.7}$$

where

$$f^i = g^{ij}\left(\frac{\partial L}{\partial x^j} - \frac{\partial^2 L}{\partial \dot{x}^j \partial x^k}\dot{x}^k\right) \tag{4.8}$$

Now the system (4.7) is exactly the flow for the vector field $S \in \chi(TQ)$

$$S = \dot{x}^i \frac{\partial}{\partial x^j} + f^i \frac{\partial}{\partial \dot{x}^i} \tag{4.9}$$

which is called the canonical semispray of L.
In terms of canonical semi-spray S one has another characterization , namely

$$i_S w_L = dH \tag{4.10}$$

where i_s denotes the interior product with respect to S , w_L is the matrix

$$w_L = \begin{pmatrix} 0 & g^{ij} \\ -g^{ij} & 0 \end{pmatrix} \tag{4.11}$$

and H is the Hamiltonian function of L

$$H = \frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L$$

This characterization yields a well known result in the theory of time -independent Lagrangian systems.

Theorem 4.3 (Conservation of Energy) If the Lagrangian L does not depend on time explicitly then the Hamiltonian H is a first integral of S , i.e. a conservation law for Euler- Lagrange equations.

Proof We have

$$L_S H = dH(S) = i_S w_L(S) = w_L(S, S) = 0.$$

It is straightforward to see that

$$L_S\theta_L = di_S\theta_L + i_S d\theta_L = d < \theta_L, S > -i_S w_L = d(\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i) - dH = dL,$$

and ,

$$L_S w_L = -L_S d\theta_L = -dL_S\theta_L = -d(dL) = 0,$$

that is

Proposition 4.4 If the Lagrangian L is regular then the Cartan 2-form w_L is invariant for the canonical semispray S.
We are now in position to apply the results of Section 3. In this way we obtain

Proposition 4.5 Let L be a regular Lagrangian and $Y \in \chi(TM)$ be a symmetry of the canonical semispray S. If $Z \in \chi(TM)$ is a Y-pseudosymmetry for S then

$$\phi = w_L(Y, Z) \tag{4.12}$$

is a first integral of S, that is a conservation law for (4.7) . In particular, if Y and Z are symmetries for the canonical semispray S then ϕ given by (4.12) is a conservation law for S.
If $Y = \begin{pmatrix} Y \\ \bar{Y} \end{pmatrix}$ and $Z = \begin{pmatrix} Z \\ \bar{Z} \end{pmatrix}$ then substitution in (4.12) yields

$$\phi = -g_{ij}\bar{Y}^i Z^j + g_{ij} Y^i \bar{Z}^j.$$

Corollary 4.6 If the Lagrangian L is regular and $Z \in \chi(TM)$ is S-pseudosymmetry for the canonical semispray S then

$$\phi = w_L(S, Z) = L_Z H \tag{4.13}$$

is a first integral of S , that is a conservation law for the equations (4.7).

Remark 4.7 If we take Z = S , then ϕ gives by (4.13) vanishes and the method becomes ineffective . Thus to get a nonzero conservation law one must find a pseudosymmetry of S which is not a symmetry of H or to apply Proposition 4.5 by choosing Y different from S.
Having discussed the framework for Lagrangian systems we now introduce the following type of symmetry which may be used to generate first integrals for variational dynamical systems.

Definition 4.8 $X \in \chi(TM)$ is called Cartan symmetry for a Lagrange space (M, L) if

$$L_X w_L = 0 \tag{4.14}$$

and

$$L_X H = 0 \tag{4.15}$$

Proposition 4.9 The canonical spray S is a Cartan symmetry.
Let $X \in \chi(TM)$ be a Cartan symmetry. By (4.14) we have

$$dL_X \theta_L = 0,$$

that is, $L_X \theta_L$ is closed.

Definition 4.10 If $L_X \theta_L$ is exact then X is called Cartan symmetry.
A key result in symmetries and conservation laws is

Theorem 4.11(Generalized Noether Theorem) If X is an exact Cartan symmetry with

$$L_X \theta_L = df \tag{4.16}$$

then

$$P_X = J(X)L - f \tag{4.17}$$

is a first integral of S , that is for the Euler- Lagrange equations (4.7).
This result says how one may go from exact Cartan symmetries to conservation laws for Lagrangian systems.
The original Noether theorem was applicable to the case in which X is the complete lift of a vector field on M. This is denoted by X^c and is the generator of the flow $(\psi_t)_t$ given by

$$\psi_t(q, \dot{q}) = (\phi_t(q), \dot{q}(\phi_t)_{*,q}) \tag{4.18}$$

Here $(\phi_t)_t$ is the flow generated by $X \in \chi(M)$ which lifts to a flow $(\psi_t)_t$ on TM given by (4.18).

Definition 4.12 The generator of the flow $(\psi_t)_t$ denoted by X^c is called the complete lift of $X \in \chi(M)$. X is an invariant of L if

$$L \bullet \psi_t = L \qquad \forall \quad t \tag{4.19}$$

Theorem 4.13 (Noether Theorem) If $X = X^i \frac{\partial}{\partial x^i}$ is an invariant of L then X^c is an exact Cartan symmetry with f=0 and the quantity

$$P_X = \dot{X}^i \frac{\partial L}{\partial \dot{x}^i} \tag{4.20}$$

is a conservation law for the Euler Lagrange equations of L.
The conservation laws obtained from (4.20) will be called clssical.

5 Poisson Bracket on tangent bundle. From conservation laws to exact Cartan symmetries

In this section we shall elucidate the notion of Poisson bracket on TM and use it to determine symmetries form first integrals of Hamiltonian vector fields on TM. All the notation we need has already been set in the proceding sections.
It may be recalled that a Poisson manifold is a smoth manifold M equipped with a Poisson bracket satisfying the four axioms of antisymmetry, bilinearity, Leibnitz rule I and Jacobi identity.

Definition 5.1 (Leibnitz rule II) The vector field $S \in \chi(TM)$ given by (4.9) defining the dynamical system (4.7) is called canonical if the time evolution satisfies the axiom

$$\frac{d}{dt}\{g_1, g_2\} = \{\frac{d}{dt}g_1, g_2\} + \{g_1, \frac{d}{dt}g_2\} \tag{5.1}$$

Using Jacobi identity we are immediately led to

Proposition 5.2 This Leibnitz rule II is satisfied if the time evolution is given by a Hamiltonian vector field.

It may be recalled that a derivation D is a linear operation on the space of smooth functions on $\Re^m$ if it satisfies the Leibnitz rule I. From the theory of manifolds it is known that in local coordinates the expression of DF assumes the form

$$DF(x) = \sum_{i=1}^{m} a^i(x)\frac{\partial F}{\partial x^i}(x)$$

for some smooth functions $a^1, a^2, \ldots, a^m$ .This fact may be used to prove that

Proposition 5.3 For any Poisson bracket $\{,\}$ on TM , we have

$$\{F,G\} = \sum_{i,j=1}^{n}\{x^i,x^j\}\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial x^j} + \{x^i,\dot{x}^j\}(\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial \dot{x}^j} - \frac{\partial G}{\partial x^i}\frac{\partial F}{\partial \dot{x}^j}) + \{\dot{x}^i,\dot{x}^j\}\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial \dot{x}^j} \tag{5.2}$$

For application to Lagrangian systems we restrict ourselves to the case

$$\{x^i,x^j\} = 0. \tag{5.3}$$

Using Leibnitz rule II we find

Proposition 5.4 The position -velocity bracket on TM is symmetric :

$$\{x^i,\dot{x}^j\} = \{x^j,\dot{x}^i\}.$$

Denote this bracket by g^{ij} . Let us apply the Leibnitz rule II again to it :

$$\{L_S \ \ x^i,\dot{x}^j\} + \{x^i, L_S \ \ \dot{x}^j\} = L_S \ \ g^{ij}$$

or

$$\{\dot{x}^i,\dot{x}^j\} + \{x^i,f^j\} = L_S \ \ g^{ij}$$

whence on antisymmetrizing we get

$$\{\dot{x}^i,\dot{x}^j\} = -\frac{1}{2}\{x^i,f^j\} + \frac{1}{2}\{x^j,f^i\}$$
$$= g^{ik}N_k^j - g^{jk}N_k^i$$

with

$$N_k^i = -\frac{1}{2}\frac{\partial f^i}{\partial \dot{x}^k} \tag{5.4}$$

We have

$$\{F,G\} = \sum_{i,j=1}^{n} g^{ij}(\frac{\partial F}{\partial x^i}\frac{\partial G}{\partial \dot{x}^j} - \frac{\partial G}{\partial x^i}\frac{\partial F}{\partial \dot{x}^j}) + (g^{ik}N_k^j - g^{jk}N_k^i)\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial \dot{x}^j}$$

If the elements of the matrix g^{ij} satisfy the condition

$$g^{ik}\frac{\partial g^{jl}}{\partial \dot{x}^k} = g^{jk}\frac{\partial g^{il}}{\partial \dot{x}^k} \tag{5.5}$$

Then clearly the operation $\{,\}$ becomes acceleration independent . Finally

Proposition 5.5 For the Poisson bracket $\{,\}$ on n tangent bundle TM we have

$$\{F,G\} = \sum_{i,j=1}^{n} g^{ij}(\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial x^j} - \frac{\partial G}{\partial \dot{x}^i}\frac{\partial F}{\partial x^j}) - g^{il}g^{jn}w_{nl}\frac{\partial F}{\partial \dot{x}^i}\frac{\partial G}{\partial \dot{x}^j} \tag{5.6}$$

with

$$w_{nl} = \frac{1}{2}(\frac{\partial f_n}{\partial \dot{x}^l} - \frac{\partial f_l}{\partial \dot{x}^n})$$
$$g^{ln}f_n = f^l$$

Returning to symmetries and conservation laws ,a straightforward calculation provides the correspondence between exact Cartan symmetires and conservation laws.

Proposition 5.6 If $F \in C^\infty(TM)$ is a first integral for the system (4.7) then $X \in \chi(TM)$ to exact Cartan symmetry with the local expression

$$X = X^i \frac{\partial}{\partial x^i} + \bar{X}^i \frac{\partial}{\partial \dot{x}^i}$$

$$X^i = \{x^i, F\} \tag{5.7}$$

$$\bar{X}^i = \{\dot{x}^i, F\} \tag{5.8}$$

Using the expression for the Poisson bracket on TM we get

Proposition 5.7 If $X \in \chi(TM)$ is exact Cartan symmetry with conservation law $F \in C^\infty(TM)$ then

$$X^k = g^{kl} \frac{\partial F}{\partial \dot{x}^i} \tag{5.9}$$

$$\bar{X}^k = -g^{kl} \frac{\partial C}{\partial x^l} - g^{kl} g^{mn} w_{nl} \frac{\partial C}{\partial \dot{x}^m} \tag{5.10}$$

We have not found in literature the local expression of exact Cartan symmetry given above.

6 Two- dimensional isotropic harmonic oscillator

In this section we study a toy model which is illustrative of the many methods of finding symmetries and conservation laws. Consider the 2D isotropic harmonic oscillator

$$\ddot{x}^1 + x^1 = 0 \tag{6.1a}$$

$$\ddot{x}^2 + x^2 = 0 \tag{6.1b}$$

The canonical semispray of the system (6.1) is

$$S = \dot{x}^1 \frac{\partial}{\partial x^1} + \dot{x}^2 \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial \dot{x}^1} - x^2 \frac{\partial}{\partial \dot{x}^2} \tag{6.2}$$

It is straightforward to integrate $L_S F = 0$ and generate the following functions $F_1, \ldots, F_4$ which form a basis for the vector space of all the quadratic first integrals of S or of the above system of differential equations

$$F_1 = x^1 \dot{x}^2 - x^2 \dot{x}^1$$

$$F_2 = x^1 x^2 + \dot{x}^1 \dot{x}^2$$

$$F_3 = \frac{1}{2}((\dot{x}^1)^2 + (\dot{x}^2)^2 + (x^1)^2 + (x^2)^2)$$

$$F_4 = \frac{1}{2}((\dot{x}^1)^2 - (\dot{x}^2)^2 + (x^1)^2 - (x^2)^2). \tag{6.3}$$

The above dynamical system is conventionally described by the Lagrangian

$$L = \frac{1}{2}[(\dot{x}^1)^2 + (\dot{x}^2)^2] - \frac{1}{2}[(x^1)^2 + (x^2)^2] \tag{6.4}$$

A straightforward computation shows that L- invariant vector field is uniquely given by

$$X = x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1} \tag{6.5}$$

and the associated classical Noetherian conservation law is F_1. The remaining nonclassical conservation laws may be obtained using (pseudo-) symmetries or the generalised Noether theorem . It is easy to verify the following functions Y_1, Y_2, Y_3 are symmetries of S.

$$Y_1 = \dot{x}^1 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial \dot{x}^1} \tag{6.6a}$$

$$Y_2 = \dot{x}^2 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial \dot{x}^2} \tag{6.6b}$$

$$Y_3 = \dot{x}^2 \frac{\partial}{\partial x^1} + \dot{x}^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial \dot{x}^1} - x^1 \frac{\partial}{\partial \dot{x}^2} \tag{6.6c}$$

Also because S is 1- homogeneous with respect to all the variables x,$\dot{x}$, we find

$$Z = x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + \dot{x}^1 \frac{\partial}{\partial \dot{x}^1} + \dot{x}^2 \frac{\partial}{\partial \dot{x}^2} \tag{6.7}$$

is a symmetry of S.To find nonclassical first integrals of S by the method of (pseudo-) symmetries we note that

$$sw_L(S, Y_i) = 0, w_L(S, Z) = 2F_3, w_L(Y_1, Z) = F_3 + F_4,$$

$$w_L(Y_2, Z) = F_3 - F_4, w_L(Y_3, Z) = 2F_2$$

These results may also be established by application of the generalised Noether theorem to the Lagrangian (6.4) and its exact Cartan symmetries (6.6).

To go from conservation laws to exact Cartan symmetries we may use the method of Poisson Brackets on tangent bundles. For the dynamical system in question the only nonvanishing Poisson brackets are

$$\{x^1, \dot{x}^1\} = -\{\dot{x}^1, x^1\} = \{x^2, \dot{x}^2\} = -\{\dot{x}^2, x^2\} = 1 \tag{6.8}$$

All other brackets between the variables are taken to vanish . It is easy to varify that

$$X = \sum_{i=1}^{2} \{x^i, F_1\} \frac{\partial}{\partial x^i}$$

$$Y_{1(2)} = \sum_{i=1}^{2} \{x^i, F_3\} \frac{\partial}{\partial x^i} \{\dot{x}^i, F_3\} \frac{\partial}{\partial \dot{x}^i} +_{(-)} \{x^i, F_4\} \frac{\partial}{\partial x^i} +_{(-)} \{\dot{x}^i, F_4\} \frac{\partial}{\partial \dot{x}^i}$$

$$Y_3 = \sum_{i=1}^{2} \{x^i, F_2\} \frac{\partial}{\partial x^i} + \{\dot{x}^i, F_2\} \frac{\partial}{\partial \dot{x}^i}$$

It is well known that the Hamiltonian function associated with the Lagrangian (6.4) is F_3 . Now observe that F_2 and F_4 are the Hamiltonian functions of the following 2- homogeneous Lagrangians L_2 and L_4 of the system (6.1) .

$$L_2 = \dot{x}^1 \dot{x}^2 - x^1 x^2 \tag{6.9a}$$

$$L_4 = \frac{1}{2}((\dot{x}^1)^2 - (\dot{x}^2)^2 - (x^1)^2 + (x^2)^2) \tag{6.9b}$$

The latter observation is very important from the point of view of inverse problem of classical mechanics [5].

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