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(ψ, α, β)- Weak contractions For Two Maps in Partially Ordered Metric Spaces

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ABSTRACT

In this paper we have obtained some common fixed point theorems for g-monotone mapping in partially ordered complete metric spaces by using the control function i.e. altering distance function. Our work is generalizations of Choudhury et al. [4] and some others.

1. Introduction

Generalizations of the Banach contraction principle (BCP) through the concept of weak contraction in Hilbert spaces is specified by Alber et al. [1]. After this result Rhoades [2] had proved it is also effective in complete metric spaces.

The concept of control function i.e. altering distance function in metric fixed point theory is given by Khan et al. [12]. Further this function have been used in a number of fixed point results. [13–17]. Through this functions the weak contraction principle has been generalized in metric spaces in [5] and in partially ordered metric spaces in [22].

Weakly contractive maps and maps which satisfies other weak contractive conditions have been given in a lot of research works. [3–11]. Compatibility of two mappings introduced by Jungck [24]. This concept has been weakened to compatibilities of type A, type B, type C and to weak compatibility [25, 26].

Recently Choudhury et al. [4] generalized the weak contraction principle to coincidence point and common fixed point results with some examples in partially ordered metric spaces.

In this paper, we have generalize and extend the results of Choudhury et al. [4] for some common fixed point theorems for two mappings satisfying the concept of g-non-decreasing and weakly compatible mappings along with altering distance function in partially ordered complete metric spaces.

2. Preliminaries:

We start this section by some basic definitions and results which are used in sequel.

Definition 2.1: [25] Let (X, d) be a metric space. Then a pair (f, g) of self maps of X is said to be weakly compatible mappings if they commute at their coincidence points that is if $fx = gx$ for some $x \in X$ implies that$fgx = gfx$.

Definition 2.2:[12] A function ψ: $[0, \infty) \to [0, \infty)$ is called an altering distance function if the following conditions are satisfied: (i) ψ is continuous and monotonically non-decreasing. (ii) ψ (0) = 0 if and only if t = 0.

Definition 2.3:[27] Let $(X, \leq)$ be a partially ordered set and f and g be two self maps on X. Then f is said to be g-non-decreasing mapping if for $x, y \in X$, $gx \leq gy$ implies $fx \leq fy$.

Theorem 2.4:[4] Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Let $f, g: X \to X$ be such that $f(X) \subseteq g(X)$, f is a g-non-decreasing, $g(X)$ is closed and

(2. 4. 1) $\psi\big(d(fx, fy)\big) \leq \alpha\big(d(gx, gy)\big) - \beta\big(d(gx, gy)\big)$ for all $x, y \in X$ such that $gx \leq gy$.

where $\psi, \alpha, \beta : [0, \infty) \to [0, \infty)$ are such that, ψ is continuous and monotone non-decreasing, α is continuous, β is lower semi-continuous,

(2. 4. 2) $\psi(t) = 0$ if and only if $t = 0$, $\alpha(0) = \beta(0) = 0$. And

(2. 4. 3) $\psi(t) - \alpha(t) + \beta(t) > 0$ for all $t > 0$.

(2. 4. 4) If any nondecreasing sequence $\{x_n\}$ in X converges to z, then we assume $x_n \leq z$ for all $n \geq 0$.

(2. 4. 5) If there exists $x_0 \in X$ such that $gx_0 \leq fx_0$, then $f$and g have a coincidence point.

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(2. 4. 6) If $gz \leq ggz$ and f and g are weakly compatible then f and g have a common fixed point in X.

3. Main Result:

We generalize the Theorem 2.4 [4] as follows

Theorem 3.1: Let $(X, \leq)$ be a partially ordered set and let there exists a metric d in X such that (X, d) is a complete metric space. Let $f, g: X \to X$ be such that $f(X) \subseteq g(X)$, $g(X)$ is closed and f is a g-non-decreasing mappings satisfies conditions (2.4.2) - (2.4.6) and

(3.1.1) $\psi\big(d(fx, fy)\big) \leq \alpha\big(M(x, y)\big) - \beta\big(M(x, y)\big)$, where

$$M(x, y) = \max\{d(gx, gy), d(gx, fx), d(gy, fy), \tfrac{1}{2}(d(gx, fy) + d(gy, fx))\}$$

for all $x,\ y \in X$ for which $gx \leq gy$.

Proof: Let x_0 be an arbitrary point in X, such that $gx_0 \leq fx_0$. starting with the given x_0 we can choose $x_1 \in X$ such that $gx_1 = fx_0$ (since$f(X) \subseteq g(X)$). Now it is $gx_0 \leq fx_0 = gx_1$ and since f is a g-non-decreasing so we have $fx_0 \leq fx_1$. In this way we construct the sequence $\{x_n\}$ recursively as

$$fx_n = gx_{n+1} \text{ for all } n \geq 1 \qquad (3.1)$$

for which

$$gx_0 \leq fx_0 = gx_1 \leq fx_1 = gx_2 \leq fx_2 \leq \cdots \leq fx_{n-1} = gx_n \leq fx_n = gx_{n+1} \leq \cdots \qquad (3.2)$$

If any two successive terms in $\{x_n\}$ are equal then the proof is complete. Therefor let us consider that $d(fx_n, fx_{n+1}) \neq 0$, for all $n \geq 1$.

Now we will prove that

$$d(fx_n, fx_{n+1}) < d(fx_{n-1}, fx_n) \text{ for all } n \geq 1.$$

Substituting $x = x_n$ and $y = x_{n+1}$ in (3.1.1) and using (3.1), (3.2) and the monotone property of ψ, we have

$$\psi\big(d(fx_n, fx_{n+1})\big) \leq \alpha\big(M(x_n, x_{n+1})\big) - \beta\big(M(x_n, x_{n+1})\big)$$

Where

$$M(x_n, x_{n+1}) = \max\{d(gx_n, gx_{n+1}), d(gx_n, fx_n), d(gx_{n+1}, fx_{n+1}), \tfrac{1}{2}(d(gx_n, fx_{n+1}) + d(gx_{n+1}, fx_n))\}$$

$$\leq \max\{d(fx_{n-1}, fx_n),\ d(fx_{n-1}, fx_n),\ d(fx_n, fx_{n+1}), \tfrac{1}{2}(d(fx_{n-1}, fx_{n+1}) + d(fx_n, fx_n))\}$$

$$\leq \max\{d(fx_{n-1}, fx_n),\ d(fx_n, fx_{n+1}),\ \tfrac{1}{2}(d(fx_{n-1}, fx_{n+1}))\}$$

Hence

$$\psi(d(fx_n, fx_{n+1})) \leq \alpha(M(x_n, x_{n+1})) - \beta\big(M(x_n, x_{n+1})\big) \qquad (3.3)$$

Where

$$M(x_n, x_{n+1}) = \max\{d(fx_{n-1}, fx_n),\ d(fx_n, fx_{n+1}),\ \tfrac{1}{2}(d(fx_{n-1}, fx_{n+1}))\}$$

Now

$$M(x_n, x_{n+1}) = \text{either } d(fx_{n-1}, fx_n) \text{ or } d(fx_n, fx_{n+1}).$$

If $M(x_n, x_{n+1}) = d(fx_n, fx_{n+1})$ then by (3.1.1), we have

$$\psi\big(d(fx_n, fx_{n+1})\big) \leq \alpha\big(f(x_n, x_{n+1})\big) - \beta\big(f(x_n, x_{n+1})\big) \qquad (3.4)$$

Using (2. 4. 3) we have $d(fx_n, fx_{n+1}) = 0$, which is the contradiction.

Hence

$$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$$

If $M(x_n, x_{n+1}) = \frac{1}{2}(d(fx_{n-1}, fx_{n+1}))$ and let $\frac{1}{2}\big(d(fx_{n-1}, fx_{n+1})\big) > 0$ then by (3.1.1), we have

$$\psi\big(d(fx_n, fx_{n+1})\big) \leq \alpha\left(\tfrac{1}{2}\big(d(fx_{n-1}, fx_{n+1})\big)\right) - \beta\left(\tfrac{1}{2}\big(d(fx_{n-1}, fx_{n+1})\big)\right)$$

$$\leq \alpha\left(\tfrac{1}{2}\big(d(fx_{n-1}, fx_n)\big) + \tfrac{1}{2}\big(d(fx_n, fx_{n+1})\big)\right) - \beta\big(\tfrac{1}{2}\big(d(fx_{n-1}, fx_n)\big) + \tfrac{1}{2}\big(d(fx_n, fx_{n+1})\big)\big)$$

$$\leq \left(\alpha\left(\tfrac{1}{2}\big(d(fx_{n-1}, fx_n)\big)\right) - \beta\left(\tfrac{1}{2}\big(d(fx_{n-1}, fx_n)\big)\right)\right) + \left(\alpha\left(\tfrac{1}{2}\big(d(fx_n, fx_{n+1})\big)\right) - \beta\big(\tfrac{1}{2}\big(d(fx_n, fx_{n+1})\big)\big)\right)$$

Using (2. 4. 3) we have $d(fx_n, fx_{n+1}) = 0$, which is the contradiction.

Hence

$$d(fx_n, fx_{n+1}) \leq d(fx_{n-1}, fx_n)$$

Therefore the sequence $\{d(fx_n, fx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers therefore there is some $r \geq 0$ such that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = r \qquad (3.5)$$

Taking n $\to$ ∞ in (3.4) and using the lower semi continuity of β and the continuities of ψ and α, we obtain $\psi(r) \leq \alpha(r) - \beta(r)$, by condition (2. 4. 3), implies that $r = 0$ i.e.

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0 \qquad (3.6)$$

Now we claim that $\{fx_n\}$ is a Cauchy sequence. Suppose to the contrary that $\{fx_n\}$ is not a Cauchy sequence then there exists some $\varepsilon > 0$ for which we can find two sequences $\{fx_{m(k)}\}$ and $\{fx_{n(k)}\}$ of $\{fx_n\}$, $n(k) > m(k) > k$ for all $k > 0$ with

$$r_k = d(fx_{m(k)},\ fx_{n(k)}) \geq \varepsilon \qquad (3.7)$$

and

$$d(fx_{m(k)},\ fx_{n(k)-1}) < \varepsilon \qquad (3.8)$$

Now from (3.7), (3.8) and using triangle inequality we have

$$\varepsilon \leq r_k \leq d(fx_{m(k)},\ fx_{n(k)}) \leq d(fx_{m(k)},\ fx_{n(k)-1}) + d(fx_{n(k)-1},\ fx_{n(k)}) < \varepsilon + d(fx_{n(k)-1}, fx_{n(k)})$$

Taking $k \to \infty$ in the above inequality and using (3.6), we obtain

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} d(fx_{m(k)},\ fx_{n(k)}) = \varepsilon \qquad (3.9)$$

Again by triangle inequality for all $k \geq 0$, we have

$$d(fx_{m(k)-1}, fx_{n(k)-1}) \leq d(fx_{m(k)-1}, fx_{m(k)}) + d(fx_{m(k)}\ fx_{n(k)}) + d(fx_{n(k)}, fx_{n(k)-1})$$

And

$$d(fx_{m(k)}, fx_{n(k)}) \leq d(fx_{m(k)}, fx_{m(k)-1}) + d(fx_{m(k)-1}, fx_{n(k)-1}) + d(fx_{n(k)-1}, fx_{n(k)})$$

Taking $k \to \infty$ in the above inequality and using (3.6) and (3.9) we obtain

$$\lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)-1}) = \varepsilon \qquad (3.10)$$

I similarly

$$\lim_{k\to\infty} d(fx_{m(k)-1}, fx_{n(k)}) = \lim_{k\to\infty} d(fx_{n(k)-1}, fx_{m(k)}) = \varepsilon \qquad (3.11)$$

Again by (3.2), $gx_{m(k)+1} = fx_{m(k)} \leq fx_{n(k)} = gx_{n(k)+1}$ are comparable.

Putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (3.1.1) for all $k \geq 0$ we have

$$\psi\left(d(fx_{m(k)}, fx_{n(k)})\right) \leq \alpha\left(\left(M(x_{m(k)}, x_{n(k)})\right)\right) - \beta(M(x_{m(k)}, x_{n(k)}))$$

Where

$$M(x_{m(k)}, x_{n(k)}) = \max\{d(gx_{m(k)}, gx_{n(k)}), d(gx_{m(k)}, fx_{m(k)}), d(gx_{n(k)}, fx_{n(k)}), \frac{1}{2}(d(gx_{m(k)}, fx_{n(k)}) + d(gx_{n(k)}, fx_{m(k)}))\}$$
$$\leq \max\{d(fx_{m(k)-1}, fx_{n(k)-1}), d(fx_{m(k)-1}, fx_{m(k)}), d(fx_{n(k)-1}, fx_{n(k)}), \frac{1}{2}(d(fx_{m(k)-1}, fx_{n(k)}) + d(fx_{n(k)-1}, fx_{m(k)}))\}$$

Taking $k \to \infty$ in the above inequality and using (3.6), (3.10), (3.11) we have $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon)\right\} = \varepsilon$

Thus

$$\psi(d(fx_{m(k)}, fx_{n(k)})) \leq \alpha\left(\left(M(x_{m(k)}, x_{n(k)})\right)\right) - \beta(M(x_{m(k)}, x_{n(k)}))$$

Taking limit as $k \to \infty$ in the above inequality, the continuities of ψ and α and the lower semi continuity of β, we have

$$\psi(\lim_{k\to\infty} d(fx_{m(k)}, fx_{n(k)})) \leq \alpha(\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}))) - \beta(\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}))) \text{ i.e.}$$

$\psi(\varepsilon) \leq \alpha(\varepsilon) - \beta(\varepsilon)$, using (2. 4. 3), we have $\varepsilon = 0$,which is a contradiction. Therefore $\{fx_n\}$ is a Cauchy sequence and hence $\{fx_n\}$ convergent in the complete metric space (X, d) since $g(X)$ is closed and by (3.1),

$fx_n = gx_{n+1}$ for all $n \geq 1$, we have that there exists $z \in X$ for which

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = gz \qquad (3.12)$$

Now we have to prove that z is a coincidence point of f and g. Since from (3.2) we have $\{gx_n\}$ is a non-decreasing sequence in X and by (2. 4. 3), (3.12) we have

$gx_n \leq gz$ for all n (3.13)

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Putting $x = x_n$ and $y = z$ in (3.1.1) and by (3.13) we have

$$\psi\big(d(fx_n, fz)\big) \leq \alpha\Big(\big(M(x_n, z)\big)\Big) - \beta\big(\big(M(x_n, z)\big)\big)$$

Where

$$M(x_n, z) = \max\{d(gx_n, gz), d(gx_n, fx_n), d(gz, fz), \tfrac{1}{2}(d(gx_n, fz) + d(gz, fx_n))\}$$

Taking $n \to \infty$ in the above inequality and using (3.12) we have

$$\lim_{n\to\infty} M(x_n, z) = \max\{d(gz, gz), d(gz, gz), d(gz, fz), \frac{1}{2}(d(gz, fz) + d(gz, gz))\}$$
$$\leq \max\{0, 0, d(gz, fz), \frac{1}{2}(d(gz, fz))\}$$

Thus

$$\psi(d(gz, fz)) \leq \alpha(d(gz, fz))) - \beta(d(gz, fz))$$

Using (2.2.2) we have $d(gz, fz) = 0$ that is

$$fz = gz. \tag{3.14}$$

Thus f and g have coincidence point.

For the uniqueness since $\{gx_n\}$ is a non-decreasing sequence converging to gz. Then by (2. 4. 6) we have $gz \leq ggz$. Since f and g are weakly compatible maps therefore using (3.14) we have $fgz = gfz$.

Now let

$$w = fz = gz \tag{3.15}$$

Therefore

$$gz \leq ggz = gw \tag{3.16}$$

Also

$$fw = fgz = gfz = gw \tag{3.17}$$

Therefore from (3.1.1) we have

$$\psi\big(d(fz, fw)\big) \leq \alpha\Big(\big(M(z, w)\big)\Big) - \beta\big(\big(M(z, w)\big)\big)$$

Where

$$M(z, w) = \max\{d(gz, gw), d(gz, fz), d(gw, fw), \tfrac{1}{2}(d(gz, fw) + d(gw, fz))\}$$
$$\leq \max\{d(fz, fw), 0, 0, \tfrac{1}{2}(d(fz, fw) + d(fw, fz))\}$$
$$= d(fz, fw)$$

Hence

$$\psi\big(d(fz, fw)\big) \leq \alpha\Big(\big(d(fz, fw)\big)\Big) - \beta\big(\big(d(fz, fw)\big)\big)$$

Using (2.2.2) we have $d(fz, fw) = 0$ i.e. $d(w, fw) = 0$, Therefore

$$w = fw = gw \tag{3.18}$$

Thus f and g have a common fixed point.

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