An Exact Analytical Solution of Einstein's Field Equations For Anisotropic Fluid Sphere

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ABSTRACT

In this paper we have obtained an exact analytical solution of Einstein's field equations for static anisotropic fluid sphere by assuming the space time to be conformally flat and by taking a suitable form of metric potential. The solution found is free from singularities and physically reasonable. We have also found Energy density ρ, radial and tangential pressure for the solution.

1. Introduction

Exact solutions to the Einstein's field equations in closed analytic form are difficult to obtain due to high non-linearity of the equations. Thus a small number of exact solutions have been obtained. The problem of constructing a static model sphere of perfect fluid (e.g. neutron model) is usually solved by numerical methods using Tolman-Oppenheimer Volkoff [7, 11] equations with an equation of static specified. Solutions of Einstein's field equations for spherically symmetric matter distribution have been solved by Adler [1], Whitman [12], Singh and Yadav [9] and some others [3, 6, 13] using different methods and assumptions. The matter distribution is usually assumed to be locally isotropic, in case of pressure for massive objects in general relativity. However, in the last few years theoretical studies on realistic stellar model indicate that some massive objects may be locally anisotropic. In fact exact analytical solutions of Einstein's field equations are of much value in general relativity. These solutions are generally obtained by using different conditions and assumptions. One of the assumptions made for obtaining the solutions is that the space time be conformally flat. This assumption has been widely used in relativity theory [2, 4, 5, 10].

Here in this paper we have obtained an exact analytical solution of Einstein's field equations for static anisotropic fluid sphere by assuming that the space time is conformally flat and by taking a judicious choice of metric potential g_{11}. The model is physically reasonable and free from singularities. Energy density □, radial and tangential pressure have been also found and discussed.

2. The Field Equations

We take the static sphenrically symmetric metric as

(2.1) $$ds^2 = e^{\upsilon} dt^2 e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where λ and υ function of r only

The Einstein's field equations.

(2.2) $$R_j^i - \frac{1}{2} R\delta_j^i = -8\pi T_j^i$$

For the spherically symmetric line element (2.1) gives

(2.3) $$-8\pi T_1^1 = e^{-\lambda}\left(\frac{\upsilon'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2}$$

(2.4) $$-8\pi T_2^2 = -8\pi T_3^3 = e^{-\lambda}\left(\frac{\upsilon''}{2} + \frac{\lambda'\upsilon'}{4} + \frac{\upsilon'^2}{4} + \frac{\upsilon' - \lambda'}{2r}\right) - \frac{1}{r^2}$$

(2.5) $$8\pi T_4^4 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2}$$

where prime denote differentiation with respect to r.

Throughout the investigation we have taken velocity of light c and the gravitational constant K to be unity. The energy momentum tensor T_j^i is given by

(2.6) $$T_j^i = (\rho + p)u^i u_j - p\delta_j^i$$

For anisotropic fluid sphere the field equations take the form

(2.7) $$-8\pi\rho = e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) - \frac{1}{r^2}$$

(2.8) $$-8\pi p_r = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right)$$

(2.9) $$-8\pi p_\perp = e^{-\lambda}\left[\frac{\upsilon' x'}{4} - \frac{2\upsilon^2}{4} - \frac{\upsilon''}{2} - \left(\frac{\upsilon' - \lambda'}{2r}\right)\right]$$

Where □ is the energy density, p_r and $p_\perp$ are the radial and tangential pressure respectively

The non-zero components of Weyl tensor for the metric (2.1)

are

(2.10) $$\begin{cases} C_{1212} = \frac{r}{12}(\upsilon' + \lambda' + r\upsilon'') - \frac{1}{6}\left(e^{\lambda} - 1\right) - \frac{r^2}{24}\left(\upsilon'\lambda' + \upsilon'^2\right) \\ C_{1313} = \sin^2\theta C_{1212} \\ C_{1010} = \frac{2e^{\upsilon}}{r^2} C_{1212} \\ C_{2323} = -2\sin^2\theta e^{-\lambda} r^2 C_{1212} \\ C_{2020} = e^{\upsilon-\lambda} C_{1212} \\ C_{3030} = -\sin^2\theta e^{\upsilon-\lambda} C_{1212} \end{cases}$$

We assume that the space time is conformally flat for which vanishing of Weyl tensor gives

(2.11) $$\frac{e^{\lambda}}{r^2} + \frac{\upsilon'\lambda'}{4} - \frac{1}{r^2} - \frac{\upsilon'^2}{4} - \frac{\upsilon''}{2} + \frac{1}{2r}\left(\upsilon' - \lambda'\right) = 0$$

Now using the transformations

(2.12) $$H = e^{-\lambda}$$

(2.13) $$Y = e^{\upsilon/2}$$

(2.14) $$x = r^2$$

(2.15) $x4, x + 1 - 4 - 4\pi x D = 0$

(2.16) $(4Hx^2)Y, xx + (2x^2 H, x - H + 1)Y = 0$

Were $D = p_r - p_\perp$ and the subscrip x following a comma denotes differentiation with respect to x.

Equations (2.15) and (2.16) on integration yield.

(2.17) $H = e^{-\lambda} = 1 + \lambda\mu r^2 + 8\pi r^2 \int_0^r \left(\frac{p_r - p_\perp}{r} \right) dr$

(2.18) $\gamma^2 = e^{\upsilon} = r^2 \left[Ae^{f(r)} + Be^{-F(r)} \right]^2$

Where □, A and B are constants of integration and

(2.19) $F(r) = \int \frac{e^{\lambda/2}}{r} dr$

The integration constants □, A and B can be evaluated by matching the metric function given by (2.17) and (2.18) to the exterior schwarzchild solution for a mass 'm' and radius r_0 as

(2.20) $A = \frac{e^{\frac{F(r_0)}{2}}}{2r_0} \left[\left(1 - \frac{2m}{r_0} \right)^{1/2} + \frac{3m}{r_0} - 1 \right]$

(2.21) $B = \frac{e^{\frac{F(r_0)}{2}}}{2r_0} \left[\left(1 - \frac{2m}{r_0} \right)^{3/2} - \frac{3m}{r_0} + 1 \right]$

(2.22) $e^{-\lambda(r_0)} = \left(1 - \frac{2m}{r_0} \right)$

3. Solutions of the field equation:

In fact the equations (2.7) – (2.9) and (2.17) – (2.19) are three equations with four unknows $\rho, p_r, p_\perp$ and F(r) so that the system is indeterminate. In order to make the system determinate, we choose.

(3.1) $H = e^{-\lambda/2} = \frac{1 + Dr^2}{1 - Er^2}$

where D and E are constants $e^{\upsilon}, p_r, p_\perp$ and ρ can be obtained from the field equations and using (2.16) – (2.22). However, for sake of convenience, we take D = □ = E/3 which yield the solution as:

(3.2) $e^{\lambda/2} = \frac{1 + 3G \in}{1 - G \in} = 1 + \frac{4G \in}{1 - G \in}$

(3.3) $e^{\lambda/2} = \frac{4G \in (1 - G)^4 + (1 + G)(1 - 3G)(1 - G \in)^4}{[(1 - G)(1 + 3G)(1 - G \in)]^2}$

(3.4) $8\pi\rho r_0^2 = \frac{8G(3 + 2G \in + 3G^2 \in^2)}{(1 + 3G \in)}$

(3.5) $$8\pi p_r r_0^2 = \frac{16G\left[\left\{\left(\frac{(1-G)^4}{(1-3G)(1+G)}-1\right)+1\right\}\left(1-2G\in+3G^2\in^2\right)-(1-G\in)^4\right]}{\left[4G\left\{\left(\frac{(1-G)^4}{(1-3G)^4(1+G)}-1\right)+1\right\}\in+(1-G\in)^4\right](1+3G\in)^4}$$

(3.6) $$8\pi p_\perp r_0^2 = 8\pi p_r r_0^2 + \frac{16G^2\in(5+3G\in)}{(1-3G\in)^3}$$

Where $E = \frac{r^2}{r_0^2}$

$$G = \frac{1-(1-2\eta)^{1/2}}{1+3(1-2\eta)^{2/2}} \text{ and } \eta = \frac{m}{r_0}$$

4. Discussion

1. It is remarkable that the solution of Einstein's field equation obtained here is singularity free and the density of fluid sphere drops continuously from its maximum value at the centre to the value which is positive at the boundary.
2. If we choose the equation of state $p_r = p_\perp$, then in this case we obtain the well knownSchwarzchild interior solution [8].

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