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Static Spherically Symmetric Perfect Fluids in Einstein – Cartan Theory

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ABSTRACT

In this paper deals with solution of Einstein-Cartan field equations for spherically symmetric line element using Tolman's technique in three different cases. We have also discussed pressure and density for the distribution.

1. Introduction

Stimulated by the successful marriage of geometrical and physical ideas in Einstein's general theory of relativity, the great French mathematician E.Cartan suggested that a more general geometrical frame work incorporating the notion of torsion as well as Riemannian curvature might be useful in the description of a continuum of spinning particles (Cartan [6], [7]) but Cartan's suggestion lay dormant for a long time. Nearly after half a century this idea has received a strong theoretical ground (both geometrical and physical) through the investigation of various authors, (Trautman ([33], [34]); Kerlick [14], Ray Chaudhary ([26], [27]), Kuchowichz ([18], [19], [20]); Hehl ([9], [10]); Hehl et.al. [11], Prasanna [24], Kopczynski ([15], [16]); Berman [2], Kannar [13], Alder [1], Bowers [3] in the form of a viable rival theory to Einstein's theory of gravitation and which is now called the Einstein-Cartan theory.

The general theory of relativity which has been considered as "most beautiful creation of single mind" has enjoyed success wherever a test has been possible [27]. The general theory of relativity has also led under general considerations to the existence of singularities in the universe. Since the singularity is not a desirable feature for any physical theory, the question arises, is it possible to keep this beautiful theory unmolested with regard to its success but at the same time modify it so as to prevent singularities? The answer seems to be in affirmative if one considers the most natural generalization of Einstein's theory as originally suggested by Elie Cartan [6, 7] which is now known as Einstein – cartan theory (or E-C theory). In this theory the intrinsic spin of matter is incorporated as the source of torsion of the space-time manifold. According to the relativistic quantum mechanics mass and spin are two fundamental characters of an elementary particle system. The energy momentum is source of curvature. By introducing torsion and relating it to the density of intrinsic angular momentum the Einstein-Cartan theory restores the analogy between mass and spin which extends to the principle of equivalence at least in its weak form. According to this principle the world line of a spin less test particle moving under the influence of gravitational fields only depends on its initial position and velocity but not on its mass. Since the predictions of E-C theory differ from those of general relativity only for matter field regions, therefore besides cosmology, an important application field of E-C theory is relativistic astrophysics which deals with the theories of the stellar objects like neutron stars with some alignment of spins of the constituents particles. Hence it is desirable to understand the full implication of the E-C theory for finite distributions like fluid spheres with non-zero pressure. With this view many workers have considered the problem of static fluid spheres in E-C theory (Prasanna [25], Kelick [14], Kuchowicz [18- 20], Skinner & Webb [29], Singh & Yadav [28], Yadav et al. [36-37], Trautman [34], has proposed that spin and torsion may avert gravitational singularities by considering of Friedman type of universe in the framework of Einstein Cartan theory and obtaining a minimum radius R_0 at t = 0. Isham, Salam and Strathdee [12] have shown that if one considers the Trautman model in the framework of their two-tensor theory then the minimum radius would increase substantially, giving a reasonable density for the universe in the early stages. Applying the same arguments for finite collapsing objects, Prasanna [23] has shown that it is possible to get a minimum critical mass for black holes. Having seen that the new idea regarding prevention of catastrophic collapse could have an interesting role in astrophysical situations. He has also discussed the complications of the Einstein-Cartan theory for finite distributions like fluid spheres with non-zero pressure. Also, since spin is a very important property of a particle, it is very relevant to consider its role in the study of such configurations as one may find in the interior of a star.

Hehl, Heyde and Kerlick [11] have considered the field equations of general relativity with spin and torsion U_4 theory to describe correctly the gravitational properties of matter on a macro physical level. They have shown how the singularities theorems of Penrose [22] and Hawking [8] must be modified to apply in E-C theory. Prasanna [25] has solved Einstein-Cartan field equations for a perfect fluid distributions and adopting Hehl's [9-10] approach, and Tolman's technique [32] obtained a number of solutions Arkuszewski et al. [4] describe the junction conditions in Einstein-Cartan theory. Raychaudhuri and Banerji [27] considered collapsing spheres in E-C theory and showed that it bounces at a radius greater than the Schwarzschild radius. Banerji [5] has pointed out that E-C spheres must bounce outside the Schwarzschild radius if it bounces at all. Yadav et. al. [36] have discussed static dust sphere in E-C theory by taking a suitable choice of effective density where as Singh and Yadav [28] have studied the static fluid spheres in E-C theory and obtained a solution in an analytic form by the method of quadrature. Spatially homogeneous cosmological models of Bianchi type VI & VII based on Einstein-Cartan theory were considered by Tsoubelis [35]. Som and Bedran [30] got the class of solutions that represent a static incoherent spherical dust distribution in equilibrium under the influence of spin. Mehra and Gokhroo [21] have also given a physically meaningful solutions of the field equations for static spherical dust distribution in E-C theory. Krori et al. [17] gave a singularity free solution for a static sphere in Einstein-Cartan theory. Suh [31] considering the static spherically symmetric interior solution in Einstein-Cartan theory closely compared with those in the Einstein theory of gravitation.

In this chapter considering spherically symmetric line element, we have solved Einstein-Cartan field equations using Tolman's technique [32] in three different cases. We have also find pressure and density for the distribution.

2. The field equations:

The Einstein-Cartan field equations are

(2.1) $R_j^i - \frac{1}{2}R\delta_j^i = kt_j^i$

(2.2) $Q_{jk}^i - \delta_j^i Q_{lk}^l - \delta_k^i Q_{jl}^l = kS_{jk}^i$

Where Q_{jk}^i is torsion tensor, t_j^i is the canonical asymmetric energy momentum tensor S_{ij}^k is the spin tensor, k=-8π.

Here we take a static spherically symmetric matter distribution represented by the line element.

(2.3) $ds^2 = -e^{2\lambda}dr^2 - r^2 d\theta^2 - r^2 sin^2\theta d\phi^2 + e^{2r}dt^2$

Where λ and v are functions of r alone. If θ'represents as orthonormal conframe we have then

(2.4) $\theta^1 - e^{\lambda}dr.\theta^2 = rd\theta, \theta^3 = rsin\theta d\phi, \theta^4 = e^{v}dt$

The metric (2.3) now becomes

$$ds^2 = -\{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2\}$$

So that

$$g_{ij} = diag\{-1, -1, -1, 1\}$$

Assuming that the spins of the individual particles composing the fluid are all aligned in the radical direction we get for the spin tensor S_{ij} the only independent non-zero component to be $S_{23} = K$, say. Since the fluid is supposed to be static, we have the velocity four-vector $u^i = \delta_4^i$.

Thus the non-zero components of $S_{jk}^i$are

(2.5) $S_{23}^4 = -S_{32}^4 = K$

Hence from the Cartan equation (2.2), we get for Q_{jk}^i the components

(2.6) $Q_{23}^4 = -Q_{32}^4 = -kK$

The others are zero

Using (2.6) in (2.3) we can obtain the torsion two-form $(G)^1$to be

(2.7) $(G)^1 = 0, (G)^2 = 0, (G)^3 = 0, (G)^4 = -kK\theta^2 \wedge \theta^3$

Once we have the torsion form we can use it in (2.3) along with (2.4) and solve the components of ω_l^k, which in the present case turn out to be

$$\omega_4^1 = \omega_4^1 = e^{-\lambda}\beta'\theta^4, \omega_1^2 = -\omega_2^1 = \frac{e^{-\lambda}}{r}\theta^2$$

(2.8) $$\omega_4^2 = \omega_2^4 = \frac{1}{2}kK\theta^3, \omega_1^3 = -\omega_3^1 = \frac{e^{-\lambda}}{r}\theta^3$$

$$\omega_4^3 = \omega_3^4 = -\frac{1}{2}kK\theta^2, \omega_2^3 = -\omega_3^2 = -\frac{1}{2}kK\theta^4 + \frac{cot\theta}{r}\theta^3$$

Using (2.8) in (2.4) we get the curvature form Ω_l^k to be

(2.9) $$\Omega_4^1 = \left[e^{-2\lambda}(v'' + v'^2 - \lambda' v')\right](\theta^1 \wedge \theta^4)$$

$$-\frac{kK}{r}e^{-\lambda}(\theta^2 \wedge \theta^3)$$

$$\Omega_4^2 = \left[\frac{1}{2}ke^{-\lambda}\left(K' + \frac{K}{r}\right)\right](\theta^1 \wedge \theta^4)$$

$$+\left[\frac{e^{-2\lambda}}{r}v + \frac{1}{4}k^2K^2\right](\theta^2 \wedge \theta^4)$$

$$\Omega_4^3 = \left[\frac{1}{2}ke^{-\lambda}\left(K' + \frac{K}{r}\right)\right](\theta^1 \wedge \theta^2)$$

$$+\left[\frac{e^{-2\lambda}}{r}v + \frac{1}{4}k^2K^2\right](\theta^3 \wedge \theta^4)$$

$$\Omega_2^1 = \frac{e^{-2r_0}}{r}\lambda'(\theta^1 \wedge \theta^2) - \frac{1}{2}kKe^{-\lambda}\left[v' - \frac{1}{r}\right](\theta^4 \wedge \theta^3)$$

$$\Omega_3^1 = \frac{e^{-2r_0}}{r}\lambda'(\theta^1 \wedge \theta^3) - \frac{1}{2}kKe^{-\lambda}\left[v' - \frac{1}{r}\right](\theta^4 \wedge \theta^2)$$

$$\Omega_3^2 = \left(\frac{1 - e^{-2\lambda}}{r^2} + \frac{1}{4}k^2K^2\right)(\theta^2 \wedge \theta^3) + \frac{1}{2}ke^{-\lambda}(K' + Kv')(\theta^1 \wedge \theta^4)$$

Equations (2.4) and (2.9) together give

(2.10) $R_{44}^1 = e^{-2\lambda}(v'' + v'^2 - \lambda' v')$

$$R_{424}^2 = R_{434}^3 = \frac{1}{4}k^2K^2 + \frac{e^{-2\lambda}v'}{r}$$

$$R_{212}^1 = R_{313}^1 = \frac{e^{-2\lambda}\lambda'}{r}$$

$$R_{323}^2 = \frac{1-e^{-2\lambda}\lambda'}{r^2} + \frac{1}{4}k^2K^2$$

$$R_{423}^1 = \frac{-kK}{r}e^{-\lambda}$$

$$R_{413}^2 = -R_{412}^3 = \frac{1}{2}ke^{-\lambda}\left(K' + \frac{K}{r}\right)$$

$$R_{243}^1 = -R_{342}^1 = \frac{1}{2}kKe^{-\lambda}\left(v' - \frac{1}{r}\right)$$

$$R_{314}^2 = \frac{!}{2}e^{-\lambda}(K' + Kv')$$

The Ricci tensor $R_{ij}$and scalar of curvature R are therefore given by

(2.11) $R_{11} = -e^{-2\lambda}\left(v'' + v'^2 - \lambda' v' - \frac{2\lambda^2}{r}\right)$

$$R_{22} = R_{33} = -\frac{e^{-2\lambda}}{r^2}[1 + r(v' - \lambda')] + \frac{1}{r^2}$$

$$R_{44} = -e^{-2\lambda}\left(v' + v'^2 - \lambda' v' - \frac{2v^2}{r}\right) + \frac{1}{2}k^2K^2$$

(2.12) R = $-2\left\{\frac{1}{r^2} - e^{-2\lambda}\right\}\left[\frac{1}{r^2} + v' + v'^2 - \lambda' v' + \frac{2}{r}(v' - \lambda')\right] + \frac{1}{2}kK^2$

With R_{ij}=0, i$\neq j$.

Hence the components of $R_{ij} - \frac{1}{2}Rg_{ji}$ is found to be

(2.13) $R_{11} - \frac{1}{2}Rg_{11} = \frac{1}{r^2} + e^{-2\lambda}\left(\frac{2v'}{r} + \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2$

$$R_{22} - \frac{1}{2}Rg_{22} = R_{33} - \frac{1}{2}Rg_{33} = e^{-2\lambda}\left[v'' + v'^2 - \lambda' v' + \frac{1}{r}(v' - \lambda') + \frac{1}{4}k^2K^2\right]$$

$$R_{44} - \frac{1}{2}Rg_{44} = \frac{1}{r^2} + e^{-2\lambda}\left(\frac{2\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2$$

Since we are considering a perfect fluid distribution with isotropic pressure p and matter density ρ we have for t'.

(2.14) $t_i^j = R^{jk}\{[(e + p)u_k - u^l\nabla_m(u^m S_{lk})]u_k - pe_{ki}\}$

Using (2.6) we get then the non-zero components

(2.15) $t_1^1 = t_2^2 = t_3^3 = -p. t_4^4 = \rho$

Hence the field equations (4.2.1) may be written using (4.2.13) and (4.2.15)

(2.16) $-\frac{1}{r^2} + e^{-2\lambda}\left(\frac{2\lambda'}{r} + \frac{1}{r^2}\right) + \frac{1}{4}k^2K^2 = -kp$

(2.17) $e^{-2\lambda}\left[v^{n}+v'^{2}-\lambda' v'+\frac{1}{r}(v'-\lambda')\right]+\frac{1}{4}k^{2}K^{2}=-k\rho$

(2.18) $-\frac{1}{r^{2}}-e^{-2\lambda}\left(\frac{2\lambda'}{r}-\frac{1}{r^{2}}\right)-\frac{1}{4}k^{2}K^{2}=k\rho$

The conservation laws give us the relations

(2.19) $\nabla,[(\rho+p)u']=0$ (matter conservation)

(2.20) $\nabla,(Ku')=0$ (spin conservation) and

(2.21) $\frac{dp}{dr}+(\rho+p)v'+\frac{1}{2}kK(K'+Kv')=0$

If we assume the equation of hydrostatic equilibrium to hold as in general relativity, namely

(2.22) $\frac{dp}{dr}+(\rho+p)v'=0$

We get the additional equation

(2.23) $K'+Kv'=0$

Solving for K we get

(2.24) $K=Ge^{-v}$

Where G is a constant of integrations to be determined. Setting

$k=-\frac{8\pi G}{c^{2}}$ with G = 1, c = 1 we can write the field equations as

(2.26) $8\pi p=16\pi^{2}K^{2}+\frac{1}{r^{2}}+e^{-2\lambda}\left(\frac{2\lambda'}{r}+\frac{1}{r^{2}}\right)$

(2.27) $e^{-2\lambda}\left[\left(\frac{v''}{r}-\frac{v'}{r^{2}}-\frac{1}{r^{3}}\right)-\lambda\left(\frac{2v'}{r}+\frac{1}{r^{2}}\right)+v'\left(\frac{\lambda'+v'}{r}\right)\right]+\frac{1}{r^{3}}=0$

Where K=Ge^{-v}, where G is constant

In fact we now have a completely determined system if an equation of state is specified. However, it is well known that in practice this set of equations is formidable to solve using a pre assigned equation of state, except perhaps for the case $\rho=p$, which may not be physically meaningful. Secondly, we have the question of boundary conditions to be chosen for fitting the solutions in the interior and the exterior of the state. A very interesting aspect of the Einstein-Cartan theory is that outside the fluid distribution the equations reduce to Einstein's equations for empty space, viz.,$R_{ij}=0$, since there is no spin density.

Following Hehl's approach [8, 9] if we define

(2.28) $\bar{p}=p-2\pi K^{2},\bar{\rho}-2\pi K^{2}$

We find that the equations take the usual general relativistic form for a static fluid sphere as given by

(2.29) $8\pi\bar{\rho}=-\frac{1}{r^{2}}+e^{-2\lambda}\left(\frac{2v'}{r}+\frac{1}{r^{2}}\right)$

(2.30) $8\pi\bar{\rho}=\frac{1}{r^{2}}+e^{-2\lambda}\left(\frac{2v'}{r}-\frac{1}{r^{2}}\right)$

with (2.27) remaining the same. The equation of continuity (2.21) now becomes given

(2.31) $\frac{d\bar{p}}{dr}+(\bar{\rho}+\bar{p})v'=0$

It is clear from these equations that it is the $\bar{p}$ and not the p which is continuous across the boundary r=a of the fluid sphere. The continuity of $\bar{p}$ across the boundary ensures that of v' (exp. $2v$). Further with $\bar{p}$ and $\bar{\rho}$ replacing p and ρ respectively we are assured that the metric coefficients are continuous across the boundary. Hence we shall apply the usual boundary conditions to the solution of equations (2.27), (2.29) and (2.30).

We use the boundary conditions

(2.32) $\left[e^{-2\lambda}\right]_{r=r_0}=\left[e^{2v}\right]_{r=r_0}=\left(1-\frac{2m}{r_0}\right)$ and

(2.33) $\bar{p}=0$ at r = r_0

Where r_0 is the radius of the fluid sphere and m is the mass of the fluid sphere. The total mass as observed by an external observer, inside the fluid sphere of radius a is given by

(2.34) m = $\int_0^{r_0}\rho r^{2}dr=4\pi\int_0^{r_0}\rho r^{2}dr-8\pi^{2}\int_0^{1}K^{2}(r)r^{2}dr$

Thus the total mass of the fluid sphere is modified by the correction

$$8\pi^{2}\int_0^{r_0}K^{2}(r)r^{2}dr$$

Equations (2.29), (2.30) and (2.31) are the same as obtained by Tolman [30], so we can use the same solutions for our discussion. Assuming that the sphere has a finite radius r = $r_0$for r >r_0, since the equations are$R_{ij}=0$, we have by Birkhoff's theorem the space-time metric represented by the Schwarzschild solution

(2.35)$ds^2=-\left(1-\frac{2m}{r}\right)^{-1}dr^2-r^2d\theta^2-r^2sin^2\theta d\phi^2+\left(1-\frac{2m}{r}\right)dt^2$

Where m is a constant associated with the mass of the sphere

3. Solution:

Case-I

Here we assume

(3.1) $e^{2v}=Ar$

where A is constant

Using (2.27) after calculations we get λ finally as

(3.2) $e^{2\lambda}=\left(Cr^{7/3}+\frac{4}{7}\right)^{-1}$

where C is a constant

Thus the line element is given by

(3.3)$ds^2=Ardt^2-\left(Cr^{\frac{7}{3}}+\frac{4}{7}\right)^{-1}-r^2(d\theta^2+sin^2\theta d\phi^2)$

(3.4) A=$\frac{1}{2r_0^{\frac{1}{3}}}$, C =$-\frac{r_0^{-7/3}}{14}$

The pressure and density are given by

(3.5) $8\pi p=\frac{16\pi^2G}{Ar}+\frac{1}{7r^2}\left\{1-\left(\frac{r}{r_0}\right)^{7/3}\right\}$

(3.6) $8\pi\rho=\frac{16\pi^2G}{Ar}+\frac{1}{21r^2}\left\{9+5\left(\frac{r}{r_0}\right)^{7/3}\right\}$

The spin density K is given by

(3.7) $\frac{K^2}{r_0}=\frac{2G}{r}$

Case-II

Here we assume

(3.8) $e^{2v}=\beta r^2+\alpha$

Using (2.27) after calculations we get λ finally as

$$e^{2\lambda}=\frac{2\beta r^2+\alpha}{(1+br^2)(\alpha+\beta^2)}$$

Thus the line element is given by

(3.9) $ds^2=\frac{\alpha+2\beta^2}{(1+br^2)(\alpha+\beta r^2)}dr^2-r^2d\theta^2-r^2sin^2\theta d\phi^2+(\alpha+\beta r^2)dt^2$

Using boundary conditions as discussed in section (2) the constants are found to be

(3.10) b=$-\frac{m}{r_0^3}$, $\alpha=\left(1-\frac{3m}{r_0}\right),\beta=\frac{m}{r_0^3}$

The pressure and density are given by

(3.11) $8\pi p=\frac{16\pi^2G^2}{\left(1-\frac{3m}{r_0}+\frac{mr^2}{r_0^3}\right)}+\frac{3m^2}{r_0^4}\frac{\left(1-\frac{r^2}{r_0^2}\right)}{\left(1-\frac{3m}{r_0}+\frac{2mr^2}{r_0^3}\right)}$

(3.12) $8\pi\rho=\frac{16\pi^2G^2}{\left(1-\frac{3m}{r_0}+\frac{mr^2}{r_0^3}\right)}+\frac{m}{r_0^3}\frac{\left(6-\frac{9m}{r_0}+\frac{mr^2}{r_0^3}\right)}{\left(1-\frac{3m}{r_0}+\frac{2mr^2}{r_0^3}\right)}$

$$-\frac{4m^2r^2}{r_0^6}\frac{1-\frac{mr^2}{r_0^3}}{\left(1-\frac{3m}{r_0}+\frac{2mr^2}{r_0^3}\right)^2}$$

At r = 0, we have now

(3.13) $8\pi\rho_0 = 8\pi\rho_0 + \frac{6m}{r_0^3}\left(1-\frac{2m}{r_0}\right)\left(1-\frac{3m}{r_0}\right)^{-1}$

The constant G associated with the spin density is found in terms of ρ_0 as

(3.14) $G = \frac{1}{4\pi}\left[8\pi\rho_0\left(1-\frac{3m}{r_0}\right) - \frac{6m}{r_0^3} + \frac{9m^2}{r_0^4}\right]^{1/2}$

Eliminating r^2 between p and ρ we can get the equation of state.

Case III

Here we assume

(3.15) $e^{2v} = Ar^5$

where A is constant

Solving (2.27) for λ we get

(3.16) $e^{2v} = \frac{1}{b_1r^5-4}$

Thus the line element is given by

(3.17) $e^{2\lambda} = \frac{1}{b_1r^5-4}dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) + Ar^5dt^2$

Using boundary conditions constants are found to be

(3.18) $A = r_0^{-5}\left(1-\frac{2m}{r_0}\right)$

$$b_1 = \left(5-\frac{2m}{r_0}\right)r_0^{\frac{1}{7}}$$

Pressure and density are found to be

(3.19) $8\pi p = \frac{16\pi^2G}{r^5} - \frac{25}{r^2} + 6b_1r^{\frac{-15}{7}}$

(3.20) $8\pi\rho = \frac{16\pi^2G}{Ar^5} + \frac{5}{r^2} - \frac{6}{7}b_1r^{\frac{-15}{7}}$

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