Volume-03
Issue-03
March-2018

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Some Problems on the Representation of Multi Valued Functions

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ARTICLE DETAILS

Article History

Keywords
Upper semi-continuity, Lower semi-continuity, Conversity, Contraction

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ABSTRACT

In this paper, the notion of upper semi-continuity, lower semi-continuity, conversity, contraction, non-expansive fixed point property for multi valued functions have been extensively studied.

INTRODUCTION

We give fixed point theorems obtained by Himmelbeg, Seghal and Morriosn, Nadler, J. and others. The developments of geometric fixed point theory for multifuction were initiated by Nadler, J. and subsequently persuaded by Markin, Assad and Kirk, Browder, Riech and others. Here we discuss these developments from the basic concepts of multi valued functions.

Definition - 1. Let E and F be two sub-sets, A multi valued map or single valued map or multi valued function - T from E to F is a map that associates with any $x \in E$ a sub-set T(x) of F, called the image or the value of T at x. T is called proper if there exists at least an element $x \in E$ such that $T(x) \neq \Phi$ that is T is not constant map Φ. In this case:

$$Dom(T) = \left\{ x \,|\, \frac{E}{T}(x) \neq \Phi \right\}$$

We see that set valued map T is characterized by its graph, the sub-set of E x F defined by -

graph (T) = $\{(x, y)/y \in T(x)\}$

Also, if G is a non empty sub-set of the product space E x F, it is the graph of the set valued map T defined by -

$y \in T(x)$ if and only if $(x, y) \in G$

Here the domain of T is the projection on E of graph (T) and the image of T, the sub-set of F is defined by-

$$\mathrm{Im}(T) = \bigcup_{x \in E} T(x) = \bigcup_{x \in Dom(T)} T(x)$$

is the projection of F of graph (T).

The inverse T^{-1} of T is the set valued map from F to E as $x \in T^{-1}(y)$ if and only if $y \in T(x)$ or equivalently,

$x \in T^{-1}(y)$ if and only if $(x, y) \in$ Graph (T)

Now, we give the following relations :

Dom (T^{-1}) = Im (T), Im (T^{-1}) = dom (T)

and graph $(T^{-1}) = \{(y,x) \in F \times E\ (x, y) \in \text{Graph } T\}$

Definition - 2. A set valued map is said to be strict if -

Dom(T) = E

i.e. images T(x) are non-empty for all $y \in T$. If k be a no-empty sub-set of T and T be a strict valued map from k to F, then we extend it to the set valued map Tk from E to F defined by:

$T_k(x) = T(x)$, [$x \in E$
Φ, $x \notin k$
Where dom (T_k) is k.

We denote that a set valued map from E to F and $k \in T$ by T/k.

Definition - 3. Let P be a property (may be closed, convex, affine etc.). We say that a multi valued map satisfied the property (P) if graph (T) satisfies this property. If the images of the sets valued map are closed, convex, compact or so on, we then say that T is closed valued map, convex valued map, compact valued map or so on.

It has been observed that if T_1 and T_2 are two sets valued maps on E into F, then, for all $x \in E$,

$T_1 + T_2 \alpha T, T_1 \cup T_2, T_1 \cap T_2, T_1 \backslash T_2$

are defined as:

(i) $(T_1 + T_2)\, x = T_1(x) + T_2(x)$

(ii) $(\alpha T_1)\, x = \alpha T_1(x)$

(iii) $(T_1 \cup T_2)\, x = T_1(x) + T_2(x)$

(iv) $(T_1 \cap T_2)\, x = T_1(x) + T_2(x)$

(v) $(T_1 \backslash T_2)\, x = T_1(x) \backslash T_2(x)$

$\overline{T}$ is defined as $\overline{T} : x \to \overline{T}(x)$

Definition - 4. We now give some elementary properties of multi valued function T as follows :

(i) $T(k_1 \cup k_2) = T(k_1) \cup T(k_2)$

(ii) $T(k_1 \cap k_2) = T(k_1) \cap T(k_2)$

(iii) $T(E \backslash K) = T(E) \backslash T(k)$

(iv) If $T(k_1 \subset k_2)$, then $T(k_1) \subset T(k_2)$

We now give some examples:

Example-1. Let $T : R \to P$ by
$T(x) = x + 1$
Define $T : R \to P$ by
$T(x) = \{0\} \cup f(x)$ for each $x \in R$
Then T is point to set valued mappings.

Example-2. Let $T_a : R \to P$ be defined by $T_a(x) = [a, x]$ for a fixed $a \neq x$, $a \in R$, then T_a is a multi valued function.

Example - 3. Let $I^2 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and
$T : I^2 \to I^2$ be defined as T (x, y) is the line element in I^2 from the point (x, 0) to the point (0, y) for each $(x, y) \in I^2$, then T is a set-valued map.

Example - 4. Let I = [0, 1] and $T : I \to I$ be defined as
$T(x) = \{y \mid \overline{0} \leq y \leq x\}$ then
T is a set valued map

Example - 5. Let $K = \{f \in C[0, 1]$ f is differentiable and $f' \in C[0, 1]\}$
and $K_0 = \{f \in K \mid f(0) = 0\ [0, 1]\}$ be two sets.

Let $T : C[0, 1] \to C[0, 1]$ be defined by $T(x) = \{f\}, f^1 \in k_0$
$= \Phi, f \in k_0\ \{f\}$, then
$D(T) = k_0$ and T is a set valued map on D(T).

Example - 6. Let be a normal space and $f : E \to R$ and $\delta f : E \to E^*$ defined as
$\partial f(x) = T \in E^*(t, y - x) \leq (y) - f(x)\ \forall\ y \in E$
then ∂f is a multi valued function which is called sub gradient of f.
We see that if f(x) = x then
$\partial f(x) = \{\text{Sio } x\}$ if $x \neq 0$
$\partial f(x) = [-1, 1]$ if $x = 1$
The multi valued function has vital application in nonlinear analysis.

Example - 7. $f : E \to F$ be a single valued map then f-1 is a set valued mapping whose domain is Im (f) and it is strict set valued mapping if f is subjective and single valued when f is injective.
It has been observed that the map plays an important role when we study equations f(x) = y and to know the behaviour of the set of solutions $f^{-1}(y)$ as y ranges over F.

Example - 8. A multi valued function T can be associated with a given function f given as $f : E \to R \cup \{+\infty\}$ as follows :

$T(x) = f(x) + R, f(x) \leq \infty$
$= \Phi, f(x) = +\infty$

Example - 9. A multi valued function mapping T defined below with the help of a family of functions f (. , u) from E to F where v ranges over a set U of parameters

$T(x) = \{f(x, u)\}, u \in U$

The above types of maps are called parameterized map and used in control theory.

IMPORTANT RESULTS :

We observe that Brouwer's fixed point theorem on a Banach contraction theorem have been extended for multi valued functions.

We give some generalizations of these.

The following theorems have been given by Kakutani and K Y fan.

Theorem - A. Let k be a non-empty compact convex subset of locally convex space E (also in particular a normal linear space). If T is a upper semi-continuous mapping of k into itself and for each $x \in k$, T(x) is non-empty then there exists a point $x_0 \in k$ such that $x_0 \in T_{x0}$.

Theorem - B. Let k be a non-empty closed convex subset of a normal linear space E and f be a single valued map of k into itself, then f has a fixed point.

Kakutani proved the above theorem in case of $E = R^n$. We prove here the case when n = 1.

Theorem - 1. Let $T_{[a,b]} \to [a, b]$ be u.s.c. multi valued function and T(x) is a closed interval for each $x \in [a, b]$ then T has a fixed point.

Proof : Here we prove that for

$\in > \exists$ a point $x_0 \in [a, b]$

Such that

$S_\in(x_0) \cap T(x_0) \not\in \Phi$

If possible, suppose that this is not the case.

Then for each x, we have

$S_\in(x_0) \cap T(x_0) = \Phi$ and T (x)

Being a closed interval, can only be the right of x or to the left of x in [a, b].

Let A and B be of x in R such that T(x) is to be right of x and B be such that T(x) is to be left of x. As $a \in A$ and $b \in B$, the both sets A and B are non-empty.

Also, $A \cap B = \Phi$ and $A \cup B = [a, b]$

Moreover, if $x_1 \in A$, then as $T(x_1)$ is compact.

$x_1 + \in < \min T(x_1)$

Let λ be a number such that

$x_1 + \in < \lambda < \min T(x_1)$

But since T is u.s.c., there exists a neighbourhood $N(x_1)$ such that

$x \in N(x_1) = T(x) \subset (\lambda, b)$

and hence,

$x \in N(x_1) \cap S_\in(x_1) \Rightarrow x \in A$

Hence A is open and by symmetry B is also open, which implies that A and B determine an open portion of [a, b] which is not possible since [a, b] is connected. But this is a contradiction and hence the assertion.

Again it follows that, for each integer n, the set

$k_n = x \in X \mid B_{I/n}(x) \cap T(x) \neq \Phi$

is non empty.

Since the mapping G is such that

$G(x) = S \cap T(x_1)$ is u.s.c.

the set

$G^{-}\Phi = x \in X \mid T(x_1) = \Phi$

is open and hence

$k_n = -G^{-}\Phi$ closed

Moreover,

$k_1 \supset k_2 \supset k_3 \supset$ .... and so,

by finite intersection axiom there exists a point $x_0 \in [a, b]$ belonging to all k_n

Then $\forall\ B_{I/n}(x_0) \cap T(x_0) \neq \Phi$

Hence, $x_0 \in T(x_0)$

From this we have the following Brouwer's theorem.

Theorem - 2. If *f* is a continuous single valued mapping of [a, b] and there exists x_0 such that

$f(x_0) = x_0$ (i.e., f has a fixed point)

Now we have the following theorem for multi valued map on [a, b]

Theorem - 3. If T is a continuous multi valued mapping of [a, b] into [a, b] such that for each x_1

T (x) $\neq \Phi$ then there exist a point $x_0 \in$ [a, b] such that

$$x \in T(x_0)' \qquad x_0 = \max T(x_0)$$

Proof : It is immediate consequence of the above discussion.

It has been discussed above that KY fan theorem was proved by Kakutani in 1941 for $E \rightarrow R^n$. Further Himmelberg generated Fan's result and Sehqal and Morrison further generalized the work of Himmelberg.

We give the new version of the generalization of KY Fan result given in Theorem A.

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