Volume-03
Issue-03
March-2018

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Some Type of Weighted Frames in Banach Spaces

Ashish Kumar

Research Scholar, University Department of Mathematics, Bhupendra Narayan Mandal University, Madhepura, 852113, Bihar

ARTICLE DETAILS

Article History

Published Online: 30 March 2018

Keywords

Weighted frame, Stability, Banach frame.

2000 AMS Subject Classification:
42C15, 42A38.

ABSTRACT

In this paper we study a necessary and sufficient condition and also a sufficient condition for the stability of weighted X_d–frames. Weighted X_d-Bessel sequences, for a sequence to be weighted X_d-frame has been obtained.

1. Introduction

Fourier transform has been a major tool in analysis for over a century. It has a lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was localized time frequency representation which has this information encoded in it. In 1946 Dennis Gabor [8], filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952 the notion of frame was determined by Duffin and Schaeffer [6] in Hilbert spaces in the

following way: Let X be a Separable Hilbert space, the system of non-zero elements $\{x_n\}_n \subset X$ be called a frame in X if there exist the constants $0<A\leq B<\infty$ such that for each x X, it is valid

$$A\,||x||^2 \leq \sum_{n=1}^{\infty} |(x, x_n)|^2 \leq B||x||^2 \quad (1.1)$$

The constants A and B in (1.1) are called lower and upper frame bounds, the number $K=A/B$ is called condition coefficient of the frame $\{x_n\}_n$. In the

case, when $K=1$, $\{x_n\}_n$ is a tight frame. Development in theory of frames in Hilbert space reduced to obtaining the analogues of the known results for the Banach case. By the theory of frames [3, 6, 7, 9] we have their Banach extensions in [1, 2, 4, 5, 10]. Frames have many properties of bases but lacks a very important one, namely, uniqueness. This property of frames make them very useful in the study of function spaces, signal and image processing, filter banks, wireless and communications etc.

In the present paper, we further study weighted Xd -frames and obtain a necessary and sufficient condition. Also, a sufficient condition for stability of X_d -frame has been given and proved if a Banach spaces has a X_d -frame then it is also has a Parseval and an exact ParsevalX_d -frames.

Further, X_d -Bessel sequence has been studied and a sufficient condition in terms of X_d -Bessel sequence, for a sequence to be a X_d -frame has been given.

Definition: 2.2 ([1]) Let X_d be a BK-space and $\{w_i f_i\}_{i=1}^{\infty} \subset X^*$. The sequence $\{w_i f_i\}_{i=1}^{\infty}$ is called a weighted X_d-frame for X with lower bound A and upper bound B if $0<A\leq B<\infty$ and for every $x\in X$ one has

(i) $\{w_i f_i(x)\}_{i=1}^{\infty} \in X_d$;

(ii) $A\|x\|\leq\|\{w_i f_i(x)\}_{i=1}^{\infty}\|\leq B\|x\|$ (2.1)

When (i) and the upper inequality in (ii) hold for every $x\in X$, $\{f_i\}_{i=1}^{\infty}$ is called a X_d-Bessel sequence for X with bound A and B . The positive constants A and B, respectively, are called lower and upper frame bounds of the weighted X_d -frame.

The inequality (2.1) is called the frame inequality. It is easy to observe that frame bounds need not be unique. Further, the X_d -frame is called tight frame if it is possible to choose A and B, satisfying (2.1) with $A=B$ and normalizedtight if A=B=1. If removal of one $w_n f_n$ renders the collection $\{w_n f_n\}$ no longer a weighted X_d frame for X, then $\{w_n f_n\}$ is called an exact weighted X_d frames. The operator U and T given by

U(x)= $\{w_i f_i\}$ and $T(d_i)=\sum d_i w_i f_i$ are called the analysis operator for $\{w_i f_i\}_{i=1}^{\infty}$ and the synthesis operator for $\{w_i f_i\}_{i=1}^{\infty}$, respectively.

2. MAIN RESULTS

The following result which is referred in this paper is listed in the form of a lemma.

Lemma 3.1 ([11]) If $\{f_n\}\subset X^*$ is a X_d- frames for X with respect to X_d. Then $\{f_n\}$ is an exact if and only if $f_n\notin[f_i]_{i\neq n}$, for all n.

Proof. It is straight forward on the line of totalness of $\{f_n\}$ obtained from frame inequality of X_d-frame.

The following result show that sufficient condition for Parseval and exact weighted frames.

Theorem: 3.2 If X be a Banach space having a weighted X_d-frame. Then X has a Parseval weighted X_d- frame as well as exactParseval weighted X_d -frame.

Proof: Let $\{w_n f_n\}\subset X^*$ be a weighted X_d-frame for X with respect to X_d. Then by frame inequality, $\{w_n f_n\}$ is total over X. Therefore by Remark 7.1, in [12], there exists an associated Banach space $\{\{w_n f_n(x)\}, x\in X\}$ with norm given by

$$\|\{w_n f_n(x)\}\|=\|x\|,\quad x\in X.$$

Hence , X has a Parseval weighted X_d– frame.

Further, we may assume, without loss of generality that $\{w_n f_n\}$ is finitely linear independent. (In case $\{w_n f_n\}$ is not finitely linear independent, we can derive a subsequence $\{w_i g_i\} \subset \{w_n f_n\}$ which is finitely linear independent and total over X), Then, for each , there exists an element n™N there exists an element $x_n\in X$, Such that $w_i f_i(x_n)=0$ for i=1,2 ,3,….,n-1 and $w_n f_n(x_n)=1$, indeed if $w_i f_i(x)=0$ for i=1,2 ,3,….,n-1 would imply $w_n f_n(x)=0$, then we would have $w_n g_n\in[w_1 g_1, w_2 g_2, \ldots, w_n g_n]$. which contradicting our assumption. Define $\{w_n h_n\}\subset X^*$ by $h_1=f_1$ and $h_n=f_n-\sum_{i=1}^{n-1} w_n f_n(x_i)h_i$, n=2,3,4…. Then $w_n h_n$ is total over X such that $w_i f_i(x_j)=\delta_{ij}$. Therefore there exist an associated Banach space

X_{d1}={ $w_n fn$ (x), $x \in X$} with norm ||{ $w_n h_n$ (x)}|| =||x||, $x \in X$. Hence { $w_n h_n$ (x)} is a weighted Parseval frame for X. Further $w_n h_n$ does not belongs to $[w_i h_i]_{i \neq n}$, for all n. Hence by Lemma 3.1, {$w_n h_n$} is a weighted exact Parseval X_d-frame for X.

we prove that following result, regarding stability of weighted X_d-frame for X.

Remark: 3.3 Since {$w_n f_n$} is total on if and only if it's finite linear combinations with rational coefficients are w^* -dense in X^* and since the conjugate space of every separable Banach space is w^*-separable. Therefore, in particular, every separable Banach space X has a weighted exact Parseval X_d-frame.

The following theorem gives exactness of weighted X_d- frames.

Theorem 3.4 A weighted X_d-frame {$w_n f_n$} is an exact if there exists a sequence of non zero operator {v_n}⊂ L(X,X) such that

$$v_n^*(f)=f,\ f \in [w_i f_i]^n_{i=1} \text{ and } v_n^*(f)=0,\ f \in [w_i f_i]^\infty_{i=n+1}.$$

Proof. Let us assume that above statement is true. Then $v_1^*(w_1 f_1)= w_1 f_1$. Let x_1 be the range of v_1 and $y_1 \in X$ be such that $x_1= v_1(y_1)$ and $w_1 f_1(x_1)=1$. Then for i=1,2,3,... we have $w_i f_i(x_1)=0$. Again, let $x_2$be in the range of v_2-v_1 and $y_2 \in X$ be such that $x_2=(v_2-v_1)y_2$ and $w_i f_i(x_2)=0$. Then $w_1 f_1(x_2)=0$, continuing like this way, let $x_n$be in the range of v_n-v_{n-1} and $y_n \in X$ be such that
$x_n=(v_n-v_{n-1})y_n$ and $w_n f_n(x_n)=0$. Then for i≠n, we have

$w_i f_i(x_n)= w_i f_i(y_n)- w_i f_i(y_{n-1})$ if i<n and otherwise zero. Thus we obtain a sequence $\{x_n\} \subset X^*$ such that $w_i f_i(x_j)= \delta_{ij}$. Therefore $w_n f_n$ does not belongs to $[w_i f_i]_{i \neq n}$ for all n. hence by Lemma 3.1.
{ $w_n f_n$ } is a weighted exact X_d-frame for X.

The following theorem gives the stability for weighted X_d- frames.

Theorem 3.5 If { $w_n f_n$ }$\subset X^*$ be a weighted X_d-frames for X and let {$u_n g_n$ }$\subset X^*$ be such that {$u_n g_n(x)$}$\subset X_d$, for all $x \in X$. Then{$u_n g_n$ } is a weighted X_d-frames for X if and only if there exist a constant K>0 such that

$$\|\{(w_n f_n - u_n g_n)(x)\}\| \leq K \text{ Min } \{\|w_n f_n(x)\|, \|u_n g_n(x)\|\},\ x \in X.$$

Proof. Let { $w_n f_n$ } and {$u_n g_n$ } are weighted X_d- frames with frame bounds are A_f, B_f, A_g , B_g respectively. The by frame inequalities we have

$$\|\{(w_n f_n - u_n g_n)(x)\}\| \leq (1+B_g/A_f)\|w_n f_n(x)\|, \text{ for all } x \in X. \quad (3.1)$$

Similarly, we have

$$\|\{(w_n f_n - u_n g_n)(x)\}\| \leq (1+B_f/A_g)\|u_n g_n(x)\|, \text{ for all } x \in X. \quad (3.2)$$

Choosing $K= (1+B_g/A_f)$ or $(1+B_f/A_g)$.

Hence we have,

$$\|\{(w_nf_n - u_ng_n)(x)\}\| \leq K \text{ Min } \{\|w_nf_n(x)\|, \|u_ng_n(x)\|\}, \; x \in X.$$

Conversely, let C_f and D_f are bounds for the weighted X_d-frame $\{w_nf_n\}$ in X^*. Then for all $x\in X$, we have

$C_f\|x\| \leq \|\{w_nf_n(x)\}\| \leq \|\{(w_nf_n - u_ng_n)(x)\}\| + \|\{u_ng_n(x)\}\|$

$\leq (1+K)\|\{u_ng_n(x)\}\|$

$\leq (1+K)(\|\{(w_nf_n - u_ng_n)(x)\}\| + \|\{w_nf_n(x)\}\|)$

$\leq (1+K)^2\|\{(w_nf_n)(x)\}\|$

$\leq (1+K)^2 D_f\|x\|$.

Therefore, we have

$$C_f/(1+K)\|x\| \leq \|\{u_ng_n(x)\}\| \leq (1+K)D_f\|x\|, \text{ for all } x\in X.$$

Hence, $\{u_ng_n\}$ is a weighted X_d-frame for X.

This completes the proof.

Acknowledgment: The author would like to express their sincere thanks to anonymous referee for valuable constructive comments and suggestions which improved this paper.

REFRENCES:

[1] Casazza P.G., Cristenson O. and Stoeva D.T., Frames expansions in separable Banach spaces, J. Math., Anal. Appl., 307: 2 (2005), 710-723.
[2] Casazza P.G., Han D. and Larson D.R. Frames for Banach spaces, Contemp. Math., 247 (1999), 149-182.
[3] Christensen O., An itroduction to frames and Riesz bases, Appl. Numer. Harmon. Anal., Birkhuser Boston, MA-(1992).
[4] Christensen O. and Stoeva D.T., -Frames in separable Banach spaces, Adv. Comput. Math., 18:2 (2003), 117-126.
[5] Christensen O. and Heil C., Perturbation of Banach frames and atomic decompositions, Math. Nachr, 185 (1997), 33-47.
[6] Duffin R.J. and Schaeffer A.C., A class of non-harmonic Fourier series, Trans. Amer. Math. Soc., 72 (1952), 341-366.
[7] Feichinger H.G. and Gröchenig K., A unified approach to atomic decompositions via integrable group representations, In: Proc. Conf. ``Function Spaces and Applications'', Lecture Notes in Math. 1302, Berlin - Heidelberg - New York, Springer (1988), 52-73.
[8] Gabor D., Theory of communications, J.I.E.E (London), 93 (3) (1946), 429-457.
[9] Gröchenig K., Describing functions: Atomic decompositions versus frames, Monatsh. Math., 112 (1991), 1-41.
[10] Han D. and Larson D.R., Frames, bases and group representation for Banach space, Memoirs, Amer. Math. Soc., 147:697 (2000), 1-91.
[11] Jain P.K., Kaushik S.K. and Vashisht L.K., On stability of Banach frames, Bull. Korean Math., 37 (112) (2001), 1-41.
[12] Singer I., Bases in Banach spaces-II, Springer, New York, 1970.