RESEARCH REVIEW JOURNALS
Volume-04
Issue-05
May-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

Uniquness and Existence of Fixed Point in Complete Pseudometric Space

Sangeeta

Department ofMathematics, MaharshiDayanand University, Rohtak, Haryana -124001, INDIA

ARTICLE DETAILS

Article History
Published Online: 15 May 2019

Keywords
Complete metric space, Banach fixed point, Continuous mapping.

ABSTRACT

In this paper, a few theorems concerning the existence and uniqueness of fixed point in complete pseudometric space are built up. The aftereffects of the continuous proviso we came to are presented and a portion of the outcomes we acquired are flowed.

1. Introduction

Fixed point theory stays an extremely incredible and polar device in current science particularly in presence and uniqueness contemplations. The principal results identified with this theme showed up in the year 1922 by the Polish mathematician S. Banash[1]. The fixed point theory of mapping has been created in metric space by numerous creators (see [4,5,13]). Further, numerous arithmetic took a shot at a contractionmappings and Banach contraction guideline (see [3,9,12]).

R. Kannan in11 demonstrated that if T is a self-mapping of complete pseudomeric space X into itself fulfilling

$(Tx, Ty) \leq [(Tx, x) + (Ty, y)]$ for all $x, y \in X$where $\alpha \in (0,1/2)$. (1.1)

Then Thas unique fixed pointinand B. Fisherproved the result with$d(Tx, Ty) \leq \alpha[d(Tx, y) + d(Ty, x)]$ for all $x, y \in X$where $\alpha \in (0,1)$. (1.2)

A similar conclusion was also obtained by R. Bhardwaj *et al.* [2], S. K. Chaterjee [6], G.E. Hardy *et al.* [10]and D. P. Shukla *et al.* [14].

In this paper we will prove some theorems on the existence and unique of the fixed point in complete pseudometric space.

2. Main Results

Theorem 2.1 Let Xbe a complete pseudometric space and let $f: X \to X$a continuous self-mapping on X, suppose $f$satisfying the condition

$d(f(x), f(y)) \leq m_1 d(x, f(x)) + m_2 d(y, f(y)) + m_3 d(x, f(y)) + m_4 d(x,y)$ (2.1)

for all $x, y \in X$, $x \neq y$and for some $m_1, m_2, m_3, m_4 \in [0,1)$ suchthat$\sum_{i=1}^{4} m_i < 1$. Then$f$has a unique fixed point.

Proof. Let$x_0$be an arbitrary point in $X$and$\{x_{n-1}\}$be the sequence of iterations for$f$at x_0, such that $(x_{n-1}) = x_n$ (2.2)

Weletthat $x_{n-1} \neq x_n$for all$n \in N$.Therefore $(x_{n-1}, x_n) = ((x_{n-2}), (x_{n-1}))$. So, $(x_{n-1}, x_n) \leq m_1(x_{n-2}, f(x_{n-2})) + m_2 d(x_{n-1}, f(x_{n-1})) + m_3 d(x_{n-2}, f(x_{n-1})) + m_4 d(x_{n-2}, x_{n-1})$. And by (2.2) we find that,$(x_{n-1}, x_n) \leq m_1(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_n) + m_4 d(x_{n-2}, x_{n-1})$.

By triangle inequality for some $x_{n-2} \leq x_{n-1} \leq$, we obtained

$(x_{n-1}, x_n) \leq m_1(x_{n-2}, x_{n-1}) + m_2 d(x_{n-1}, x_n) + m_3 d(x_{n-2}, x_{n-1}) + m_3 d(x_{n-1}, x_n) + m_4 d(x_{n-2}, x_{n-1})$.

$$= \left(\frac{m_1+m_3+m_4}{1-m_2-m_4}\right)(x_{n-2}, x_{n-1})$$

And, $(x_{n-1}, x_n) \leq \left(\frac{m_1+m_3+m_4}{1-m_2-m_3}\right)^2 (x_{n-3}, x_{n-1})$. So, if we repeat this work we obtain$(x_{n-1}, x_n) \leq \left(\frac{m_1+m_3+m_4}{1-m_2-m_3}\right)^n (x_0, x_1)$

For some $s \geq n-1$, we have

$d(x_{n-1}, x_s) \le d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + \cdots + d(x_{s-1}, x_s)$ by (2.2) we conclude that$d(x_{n-1}, x_s) \le \{\beta^n + \beta^{n+1} + \cdots + \beta^s\} d(x_0, x_1)$, where $\beta = \left(\frac{m_1+m_3+m_4}{1-m_2-m_3}\right)$, and since$\sum_{i=1}^{4} m_i < 1$. Therefore $\to 0$ $as n \to \infty$. Then, $(x_{n-1}, x_s) \to 0$ $as s \to \infty$.

EveryCauchysequence $\{x_{n-1}\}$ in $X$is convergence, since $X$is a complete space.i.e.there exist $z_1 \in X$such that $x_n \to z_1$, also we have a continuous self-mapping, then$f(\lim_{n\to\infty} x_n) = f(z_1) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = z_1$

Hence, $z_1$is a fixed point of $f$in X.

We need to prove that $z_1$is a unique fixed point of $f$in X, for that we let that there exists another fixed point $z_2 \in X$, such that $z_1 \ne z_2$and $f(z_1) = z_1$, $f(z_2) = z_2$, therefore by (2.1),$d(z_1, z_2) = d(f(z_1), f(z_2)) \le m_1 d(z_1, f(z_1)) + m_2 d(z_2, f(z_2)) + m_3 d(z_1, f(z_2)) + m_4 d(z_1, z_2)$.$(z_1, z_2) \le (m_3 + m_4)(z_1, z_2)$, $(m_3 + m_4) < 1$.

Then $(z_1, z_2) \le 0$, which implies that $(z_1, z_2) = 0$, so $z_1 = z_2$and$z_1$is a unique fixed point of $f$in$X$.

Theorem 2.2 Let Xbe a complete pseudometric space and let f_1, $f_2$two continuous self-mappings on$X$, satisfying the condition

$$d(f_1(x), f_2(y)) \le m_1 d(x, f_1(x)) + m_2 d(y, f_2(y)) + m_3 d(x, f_2(y)) + m_4 d(x,y) \quad (2.3)$$

for all $x, y \in X$, $x \ne y$and for some $m_1, m_2, m_3, m_4 \in [0,1)$ suchthat $\sum_{i=1}^{4} m_i < 1$. Then$f_1$and $f_2$have a unique common fixed point.

Proof. For $x_0 \in X$, $y_0 \in X$take $f_1(x_{k-1}) = x_n$, $f_2(y_{k-1}) = y_n$. So,$d(x_k, y_k) = d(f_1(x_{k-1}), f_2(y_{k-1})) \le m_1 d(x_{k-1}, f_1(x_{k-1})) + m_2 d(y_{k-1}, f_2(y_{k-1})) + m_3 d(x_{k-1}, f_2(y_{k-1})) + m_4 d(x_{k-1}, y_{k-1})$.

$= m_1(x_{k-1}, x_k) + m_2 d(y_{k-1}, y_k) + m_3 d(x_{k-1}, y_k) + m_4 d(x_{k-1}, y_{k-1})$.

$(x_k, y_k) \le\le [m_1(x_0, x_n) + m_2 d(y_0, y_n) + m_3 \sum (x_{k-1}, y_k)$

Also, $(x_k, x_{k+1}) \le (m_1 + m_4)(x_0, x_n) + (m_2 + m_3) d(x_1, x_{n+1})$

Hence, $\sum (x_k, x_{k+1}) < \infty$. (2.4)

This implies that $(x_k, x_{k+1}) \to 0$ as $k \to \infty$, therefore we see that $\{x_k\}$ is a Cauchy sequence in X. Similarly, we can show that $\{y_k\}$ is a Cauchy sequence in X, and since $X$is a complete pseudometric space, so there exists a common fixed point in X,

Let $z_1 = \lim x_n, z_2 = \lim y_n$, for all $z_1, z_2 \in X$.

This implies that

$(x_n, z_1) \to 0$, $\to \infty$

$(y_n, z_2) \to 0$, $\to \infty$

As f_1, $f_2$are continuous mapping, we get

$(f_1(x_n), f_1(z_1)) \to 0$, $n \to \infty$

$(f_2(y_n), f_2(z_2)) \to 0$, $n \to \infty$

This means that

$d(z_1, f_1(z_1)) = d(f^{-1}(f_1(z_1), f_1(z_1)) \le m_1 d(f_1(z_1), f^{-1}(f_1(z_1)) + m_2 d(z_1, f_1(z_1)) + m_3 d(f_1(z_1), f_1(z_1)) + m_4 d(f_1(z_1), z_1)$ (2.5)

$= m_1(f_1(z_1), z_1) + m_2 d(z_1, f_1(z_1)) + m_4 d(f_1(z_1), z_1)$.

$= (m_1 + m_2 + m_4)(z_1, f_1(z_1))$.

Hence, $f_1(z_1) = z_1$. Similarly, we can show that $f_2(z_2) = z_2$.

Now, for each $m_1, m_2, m_3, m_4 \in [0,1)$ and $z_1, z_2 \in X$, we get

$d(z_1, z_2) = d(f_1(z_1), f_2(z_2)) \le m_1 d(z_1, f_1(z_1)) + m_2 d(z_2, f_2(z_2)) + m_3 d(z_1, f_2(z_2)) + m_4 d(z_1, z_2)$.

$(z_1, z_2) \le m_1(z_1, z_1) + m_2 d(z_2, z_2) + m_3 d(z_1, z_2) + m_4 d(z_1, z_2)$.

$= (m_3 + m_4)(z_1, z_2)$

Therefore, $(z_1, z_2) \le 0$.

Hence, $z_1 = z_2$. This implies that $z_1$is common fixed point of $f_1$and $f_2$in X. Now, let $z_3 \in X$be another fixed point of $f_1$and $f_2$in Xsuch that

$f_1(z_3) = z_3$and $f_2(z_3) = z_3$.

Therefore,

$d(z_1, z_3) = d(f_1(z_1), f_2(z_3)) \le m_1 d(z_1, f_1(z_1)) + m_2 d(z_3, f_2(z_3)) + m_3 d(z_1, f_2(z_3)) + m_4 d(z_1, z_3)$.

$(z_1, z_3) \leq m_1(z_1, z_1) + m_2d(z_3, z_3) + m_3d(z_1, z_3) + m_4d(z_1, z_3)$.
$= (m_3+m_4)(z_1, z_3)$.

We get that $(z_1, z_3) \leq 0$. This implies that $z_1 = z_2 = z_3$. Thus z_1 is the unique fixed point of f_1 and f_2 in X.

In the next theorem, we will generalize Theorem 2.1 and 2.2.

Theorem 2.3 Let $\{f_\propto\}_{\propto\in\Delta}$ be a family continuous self-mapping in complete pseudometric space X, suppose that

$$d(f_\propto(x), f_\beta(y)) \leq m_1d(x, f_\propto(x)) + m_2d(y, f_\beta(y)) + m_3d(x, f_\beta(y)) + m_4d(x,y) \qquad (2.6)$$

for every $x, y\in X$, $x \neq y$ and $m_1, m_2, m_3, m_4 \in [0,1)$, $\sum_{i=1}^{4} m_i < 1$. Then there exist a unique $z_1\in X$ satisfies $f_\propto(z_1) = z_1$ for all $\propto\in \Delta$.

Proof. If we repeat the same work in Theorem 2.2 but we must replace f_1 and f_2 by $f_\propto$ and f_β respectively, we will get a unique point $z_1\in X$ which satisfies $f_\propto(z_1) = f_\beta(z_1) = z_1$.

In the next theorem we will study the existences and uniqueness of a common fixed point of two mappings which are not necessarily continuous.

Theorem 2.4 Let f_1 and f_2 be two self-mappings on a complete pseudometric space X satisfies $d(f_1(x), f_2(y)) \leq m_1d(x, f_1(x)) + m_2d(y, f_2(y)) + m_3d(x, f_2(y)) + m_4d(x, y)$ for all $x, y\in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ suchthat $\sum^4$ $m_i < 1$. Supposethat $f_1f_2 = f_2f_1$ is continuous then f_1 and f_2 having unique common fixed point in .

Proof. Take $x_n = f_1(x_{n-1})$, $x_n = f_2(x_{n-1})$ and $f_1(x_{n-1}) \neq f_2(x_{n-1})$, $x_n \neq x_{n-1}, \forall n\in N$.

Therefore, $d(x_{2n+1},x_{2n}) = d(f_1(x_{2n}),f_2(x_{2n-1})) \leq m_1d(x_{2n},f_1(x_{2n})) + m_2d(x_{2n-1},f_2(x_{2n-1})) + m_3d(x_{2n},f_2(x_{2n-1})) + m_4d(x_{2n},x_{2n-1})$. (2.7)

$= m_1(x_{2n}, x_{2n+1}) + m_2d(x_{2n-1}, x_{2n}) + m_3d(x_{2n}, x_{2n}) + m_4d(x_{2n}, x_{2n-1})$.

$(x_{2n+1}, x_{2n}) \leq (x_{2n},x_{2n-1})$ (2.8) by repeating this work

since x_n is a Cauchysequencelet, $x_n \to z_1$, $n \to\infty$.

Then, $x_{n_k} \to z_1$, $k \to \infty$.

So, we have, $f_1f_2(z_1) = f_2f_1(z_1) = f_1f_2(\lim x_{n_k}) = \lim x_{n_{k+1}} = z_1$.

Let z_1 is a fixed point of f_1f_2 in X i.e. $f_1f_2(z_1) = z_1$, so we must show that $f_1(z_1) = z_1$ and $f_2(z_1) = z_1$.

Further suppose that $f_1(z_1) \neq z_1$ and $f_2(z_1) \neq z_1$,

So, $d(z_1, f_1(z_1)) = d(f_2f_1(z_1), f_1(z_1)) \leq m_1d(f_1(z_1), f_2f_1(z_1)) + m_2d(z_1, f_1(z_1)) + m_3d(f_1(z_1), f_1(z_1)) + m_4d(f_1(z_1), z_1) = 0$.

Hence, z_1 is a fixed point of f_1 in X. Also we can get that z_1 is a fixed point of f_2 in X. Therefore, f_1 and f_2 have a common fixed point which is $z_1\in X$. Now, we need to prove the common fixed point z_1 is a unique. Let $z_2\in X$, $z_2\neq z_1$ be another fixed point of f_1 and f_2 such that $f_1(z_2) = z_2$ and $f_2(z_2) = z_2$, weget

$d(z_1, z_2) = d(f_1(z_1), f_2(z_2)) \leq m_1d(z_1, f_1(z_1)) + m_2d(z_2, f_2(z_2)) + m_3d(z_1, f_2(z_2)) + m_4d(z_1, z_2)$.
$= m_1(z_1, z_1) + m_2d(z_2, z_2) + m_3d(z_1, z_2) + m_4d(z_1, z_2)$.
$= (m_3 + m_4)(z_1, z_2)$.

So, $(z_1, z_2) = 0$. Hence, z_1 is a unique common fixed point of f_1 and f_2.

Theorem 2.5 Let f_n be a self-mapping on a complete pseudometric space X, with z_n fixed point forall $z_n\in X$, $\forall n$ respectively, such that

$$d(f_n(x), f_n(y)) \leq m_1d(x, f_n(x)) + m_2d(y, f_n(y)) + m_3d(x,f_n(y)) + m_4d(x,y) \qquad (2.9)$$

forall $x, y\in X$, $x \neq y$ and for some $m_1, m_2, m_3, m_4 \in [0,1)$ such that $\sum m_i < 1$. If $(z_1) = z_1$ and $f_n \to f$ then $(z_1) = z_1$, $\forall n$.

Proof. We need to prove that $z_n \to z_1$, $\forall n$. Therefore,

$d(z_n, z_1) = d(f_n(z_n),f(z_1)) \leq m_1d(z_n, f_n(z_n)) + m_2d(z_1, f_1(z_1)) + m_3d(z_n, f_1(z_1)) + m_4d(z_n,z_1) = (m_3 + m_4)(z_n, z_1)$. (2.10)

So, $(z_n, z_1) = 0$, Therefore, $z_n = z_1, \forall n$.
Hence, $z_1$is a unique fixed point of on$X$.

REMARK:

1. If $m_1 = m_2 = m_3 = m_4 = 0$, then $x =$, but this is contradiction withTheorem 2.1.
2. If $m_1 = m_2 = m_3 = 0$, then Theorem 2.1 tends to S.Banach [1].
3. If $m_3 = m_4 = 0$, then Theorem 2.1 tends to R.Kanan [11].
4. If $m_3 = 0$, then Theorem 2.1 reduce to D. P.Shukla [14].
5. If we add $(m_5 d(y, f(x))$ such that $m_5 \in [0,1)$ and$\sum m_i < 1$, to the right sideof(2.1), then Theorem 2.1 changes to G. E. Hardy [10].
6. If $m_1 = m_2 = m_4 = 0$, then 5 tends to B. Fisher [7].

3. Application

Gopal *et al*. [8] introduced the Banach's fixed point theorem as the following:

Definition 3.1 Let (X, d) be a metric and let $f: X \to X$be a mapping

a) A point $x \in X$is called a fixed point of $f$if $x = fx$.

b) $f$is called a contraction if there exists a fixed constant $\alpha < 1$ such that: $(fx, fy) \leq (x, y)$ for all $x, y \in X$. (3.1)

Theorem3.1 Let (X, d) be a completepseudometric space and $f: X \to X$ be a contraction, for example f fulfills (3.1). At that point there exists a unique fixed point.

Along these lines, on the off chance that we think about the condition (3.1) with the condition (2.1) we get that condition (3.1) infers that the continuality of the mapping is acknowledged by [8] yet (2.1) isn't fundamental. Along these lines, (2.1) and (3.1) are totally independent.

Example 3.1 Let $X = [0,1]$ and $f(x_1) = x/3$, $x \in [0,1/3)$ $f(x_2) = x/4$, $x \in (1/3 , 1]$, $f(x)$is discontinuous at x =1. So (3.1) is not true since,

$$((x_1), (x_2)) \leq (x_1, x_2)$$

$$\Longrightarrow |x/3 - x/4| \leq \alpha |x_1, x_2|$$

$$\Longrightarrow |x/12| \leq \alpha |x_1, x_2|$$

Take $\alpha = 1/3$ and $x_1 \in [0, 1/3)$,$x_2 \in (1/3, 1]$ we get that $x \notin [0,1]$ $\forall x_1 \in [0,1/3)$, $x_2 \in (1/3,1]$and$\forall \alpha < 1$.Ontheotherhand,thecondition(2.1)issatisfiedforall$m_1, m_2, m_3, m_4 \in (0,1)$and $x_1 \in [0,1/3)$, $x_2 \in (1/3 , 1]$. Hence, Theorem 2.1 is true with a unique fixed point whichis$x = 0$, such that the fixed point of (x_1) and (x_2) need not have a common fixed point if$X = [0,1]$ by D. R. Smart [15].

References

[1]. S. Banach, Sur Les operations' dand Les ensembles abstraitetLeur application aux equations, integrals *Fundam. Math.*, (3)133-181,(1922).

[2]. R. Bhardwaj, S. S. Rajput and R. N. Yadava, Application of fixed point theory in metric spaces, *Thi Journal of Mathematics* (5)253-259,(2007).

[3]. B. ̄ ̈ĆiĆAgeneralization of banach'scontraction principle, *American Mathematical Society* (45)267-273,(1974).

[4]. A.K.Chanbey,D.P.Shan,FixedpointofcontractiontypemappinginBanachspace,*Napier Ind. Adv. Res. J. of Science*, (6)43-46, (2011).

[5]. B.S.Chandhary,uniquefixedpointtheoremforweaklyC-contractivemapping,*Kathmandn. Univ. J. of Science, Engg. And Tech.*, 5(1)6-13, (2009).

[6]. S.K.Chatterjee,Fixedpointtheorems.*Comtes.Rend.Acad.BulgariaSci.*(25)727-730,(1972).

[7]. B. Fisher, A fixed point theorem for compact metric space, *Publ. Inst Math*, (25)193- 194, (1976).

[8]. D.Gopal,D.K.PatelandS.Sukla,Banachfixedpointtheoremanditsgeneralizations,*Taylor and Francis Group*, p2, (2018).

[9]. S. Goyal, A review article on some generalizations of Banach's contractionprinciple, *IRJET*, 3(3)641-647, (2016).

[10]. G. E. Hardy and T. D. Rogers, A Generalization of Fixed Point Theorem ofReich, *Canada. Math. Bull*. Vol. 16(2), 201-206, (1973).

[11]. R. Kannan, some results on fixed points-II. *Bulletin of Calcutta Math. Soc*. (60) 71-76, (1969).

[12]. J.MerryfieldandD.Steinjr,AgeneralizationoftheBanachcontractionprinciple, *J.Math. Anal. Appl*. (237)112-120,(2002).

[13]. D. P. Shukla, S. K. Tiwari, Fixed point theorem for weakly s-contractive mappings, *Gen. Math. Notes*, 4(1)28-34,(2011).

[14]. D.P.Shukla,S.TiwariandS.Pandey,uniquefixedpointtheoremsinmetricspace, *IJIET*(3)192-197, (2013).

[15]. D. R. Smart, Fixed point theorems, Cambridge University, Press(1974).

