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Origin of Certain Generating Functions of Confluent Hypergeometric Polynomials from a Lie Point of Lie Algebra

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ABSTRACT

In this paper, new generating functions are obtained in the case of Confluent Hypergeometric polynomials, ${}_1F_1(n;\alpha;x)$ from a view point of Lie-algebra by interpreting α suitable. Also new generating functions are obtained as particular cases.

1. Introduction

The Confluent Hypergeometric polynomials ${}_1F_1(n;\alpha;x)$ satisfy the following ordinary differential equation

$$ {}_xD^2\,{}_1F_1(n;\alpha;x)+(\alpha-x)D\,{}_1F_1(n;\alpha,x)-n_1+_1F_1(n;\alpha;x)=0 \tag{1.1}$$

In a recent paper Panja and Basu (1999) has obtained some generating function for ${}_1F_1(n;\alpha,x)$

Interpreting α suitably with the help of Weisner (McBride 1971) method. Aim of the present paper is to apply Miller (Pontrjagin) method to find certain generating function for ${}_1F_1(n;\alpha,x)$ by giving suitable interpretation to α.

2.Lie-group- theoretic discussion. Let G(0, 1) be a complex 4-dimensional Lie group. The abstract group G(0, 1) consists of all 4x 4 matrices of the form.

$$ g=\begin{pmatrix} 1 & Ce^{\tau} & a & \tau \\ 0 & e^{\tau} & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};a,b,c,\tau\in C \tag{2.1}$$

where the group operation is matrix multiplication. We can introduce Co-ordinates for the elements g of G(0, 1) by setting g = (a, b, c, τ). The Co-Ordinates are valid over the entire group. The usual topology of C induces a topology in G(0, 1) and is simply connected (Pontrajagin 1958, Chapter 8). L[G(0, 1)] can be identified with the spaces of 4 × 4 matrices of the form

$$ \alpha=\begin{pmatrix} 0 & x_2 & x_4 & x_2 \\ 0 & x_3 & x_1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix};x_1,x_2,x_3,x_4\in C $$

with the Lie product $[\alpha,\beta]=\alpha\beta-\beta\alpha,\alpha\beta\in L[L(0,1)]$.

The matrices

$$g^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, g^- \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$g^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \varepsilon = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

form a basis of the Lie algebra.

L[G(0, 1)] = g(0, 1) with commutation relation.

$[\varepsilon, g^+] = [\varepsilon, g^-] = [\varepsilon, g^3] = 0$ where 0 is the 4 × 4 zero matrix.

The mapping $\alpha \to \exp \alpha$ is an analyhtic differomorphism of a neighborhood of $\theta \in L(G)$ on to a neighborhood of e in G (here θ is the additive identity of L[G(0, 1)]) where e is the identity element of G(0, 1). The mapping defines a local one to one co-ordinate transformation in C^4.

Here

$$\exp cg^3 = \begin{bmatrix} 1 & 0 & 0 & c \\ 0 & c & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \exp ag^+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & a & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\exp bg^- = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \exp d\varepsilon = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(a, b, c, d \in C)$$

Let ρ be the representative of g(0, 1) on the complex vector space V and let $J^+ = \rho(g^+)$, $J^- = \rho(g^{-1})$, $J^3 = \rho(g^3)$, $E = \rho(\varepsilon)$. There operators obey the commutation relations

$$[J^+, J^-] = E, [J^3, J^+] = J^+, [J^3, J^{-1}] = -J^-$$

$$[J^+, e] = [J^-, E] = [J^3, E] = 0$$

where [A, B] = AB – BA for the linear operators A and B on V. We define a casimir operator $C_{0,1}$ on V by

$$C_{0,1} = J^+J^- - EJ^3$$

Let ρ be the representative of g(0, 1) satisfying the conditions (i) ρ is irreducible (ii) each eigen value of J^2 has multiplicity equal to one. The countable basis for V consisting of eigen vectors of J^3.

The irreducible representation $\downarrow \omega, \mu \in C$ such that $\mu \neq 0$. The spectrum of $\downarrow \omega, \mu$ is the set S = (–ω + n; is a non negative integer) and there is a basis (f_m) $m \in S$ for the representation space V such that.

(2.2) $$J^3 f_\alpha = \alpha f_\alpha, \quad E f_\alpha = \mu f_\alpha, \; J^+ f_\alpha = \mu f_{\alpha+1}$$

$$J^- f_\alpha = (\alpha + \omega) f_{\alpha-1}, \; C_{0,1} f_\alpha = (J^+ J^- - EJ^3), \; f_\alpha = \mu\omega f_\alpha$$

where the differential operators $J^\pm, J^3$ are given by

(2.3) $$J^- = (x, \partial / \partial x + \partial / \partial y - 1) e^{-y}$$

(2.4) $$J^- = (x, \partial / \partial x - 1) e^{y}$$

(2.5) $$J^- = (x, \partial / \partial x - 1) e^{y}$$

such that

(2.6) $$J^- ({}_1F_1(n;\alpha;x) e^{\alpha y}] = (\alpha - 1) =_1 F_1(nl\alpha - 1; x) e^{(\alpha-1)y}$$

(2.7) $$J^+ [{}_1F_1(n;\alpha;x) e^{\alpha y}] = (n - \alpha) / \alpha_1 F_1(n; \alpha + 1; x) e^{(a+1)y}$$

(2.8) $$J^3 [{}_1F_1(n;\alpha;x) e^{\alpha y}] =_1 F_1(nl\alpha, x) e^{\alpha y}$$

(2.9) $$E[{}_1F_1(n;\alpha;x) e^{\alpha y}] =_1 F_1(n;\alpha;x) e^{\alpha y}$$

From the relations it follows $\mu = 1, \omega = -1$.

Thus the realization by differential operations yields a multiplier representation T of G (0, 1) whose Lie-algebra is g(0, 1).

3. Derivation of Generating functions: It follows that the function $f_\alpha(x, y) =_1 F_1(n;\alpha;x) e^{\alpha y}$ form a basis for a realization of g(0, 1) by Lie derivatives can be extended to a local multiplier representation G(0,1) defined on Cl_2, the complex vector space of all entire analytic functions Cl_2, is invariant under the given operators. The operators will uniquely define a multiplier representation T of G(0, 1) on Cl_2.

After computation of the multiplier we have

(3.1) $$T[(\exp aJ^+) f](x, t) = e^{-at} f(x + at, t)$$

(3.2) $$T[(\exp bJ^-) f](x, t) = (1 + b/t)^{-1} f(x(1 + b/t), b + t)$$

(3.3) $$T[\exp cJ^3) f](x, t) = f(x, t, e^\tau)$$

(3.4) $$T[\exp dE) f](x, t) = \exp(d) f(x, t) \text{ where } t = e^y$$

We have

$$T[g) f](x, t) = T[(\exp aJ^+)(\exp bJ^-)(\exp cJ^+)(\exp dE) f](x, t)$$

$$= T[(\exp aJ^+) T(\exp bJ^-) T(\exp cJ^+) T(\exp dE) f](x, t)$$

(3.5) $$= e^{d-at} (1 + b/t)^{-1} f[(x + at)(1 + b/t), (t + b) e^\tau]$$

where

$$g = (\exp aJ^+), (\exp bJ^-), (\exp cJ^3).(\exp dE) \in G(0,1)_0$$

The matrix elements of this local representation with respect to the basis f_α are uniquely determined by $R(1, \alpha_\theta - 1)$ and we obtain the relation

$$T[(g)f_{\alpha+k}](x,t)=\sum_{|\infty-\infty}^{\infty}A_{lk}(g)f_{\alpha+k}(x,t);k=0,\pm1\ldots \tag{3.6}$$

Using (3.5) and (3.6) and $f(x,t)={}_1F_1(n;\alpha;x)t^{\alpha}$ we can derive

$$\exp(d-at)(1+b/t)^{-1}(t+b)^{\alpha_0}+ke^{(\alpha_0+k)c}{}_1F_1(nl\alpha_0+k;(x+at)(1+b/t))$$

$$\sum_{|=\infty}^{\infty}A^{k}l,{}_{\alpha-\alpha_0}(g){}_1F_1(n;\alpha_0+l;x)t^{\alpha_0}+1 \tag{3.7}$$

where

$$A_{lk}(g)=\frac{\exp([d+(\alpha_0+k)c]\Gamma\alpha_0-1+k+1))}{(k-1)!\Gamma(\alpha_0-1+1+1)}$$

$${}_1F_1(-\alpha_0+1-1);k-1+1;ab)\ \text{if}\ k\geq k.$$

$$=\frac{\exp([d+(\alpha_0+k).c]}{(1-k)!}(a)^{1-k0}.{}_1F_1(-(\alpha_1+1-1);1-k+1;ab)\ \text{if}\ 1\geq k$$

Thus, we have the following generating functions.

$$\exp(d-at)(1+b/t)^{-1}(t+b)^{\alpha_0+k}e^{(\alpha_0+k)c}{}_1F_1(n;\alpha_0+k;(x+at).(1+b/t))$$

$$=\sum_{t=0}^{\infty}\frac{\exp[d+(\alpha_0+k).c]}{(k-1)!\Gamma(\alpha_0+1)}\Gamma(\alpha_0+k)b^{k-1}$$

$${}_1F_1(-\alpha_0-1-1);k-1+1;ab),\ {}_1F_1(n,-\alpha_0+l;x)t^{\alpha_0}+1\ \text{if}\ k\geq1. \tag{3.8}$$

and

$$\exp(d-at)(1+b/t)^{-1}(t+b)^{\alpha_0+k}e^{(\alpha_0+k)c}{}_1F_1(n;\alpha_0+k;(x+at).(1+b/t))$$

$$=\sum_{l=0}^{\infty}\frac{\exp[d+(\alpha_0+k).c]}{(1-k)}(a)^{l-k}{}_1F_1(-\alpha_0+1-k,1-k+l;ab)$$

$${}_1F_1(n,\alpha_0+l;x);t^{\alpha_0+1}\text{if}\ 1\geq k \tag{3.9}$$

(3.8) and (3.9) reduce to

$$\exp(d-at)(1+b/t)^{-1}(t+b)\alpha{}_1F_1(n,\alpha;(x+at).(1+b/t))$$

$$=\sum_{N=0}^{\infty}\frac{\Gamma(\alpha)}{\Gamma(\alpha-N)}{}_1F_1(-\alpha+N-1,N+1;ab).$$

$${}_1F_1(n,\alpha-N;x);t^{\alpha-N}\ \text{if}\ k\geq1 \tag{3.10}$$

and

$$\exp(-at)(1+b/t)^{-1}(t+b)^{\alpha}{}_1F_1(n,\alpha;(x+at).(1+b/t)).$$

$$=\sum_{N=0}^{\infty}\frac{(a)^{N}}{N!}{}_1F_1(1-\alpha;N+1;ab).{}_1F_1(n;\alpha+N,x)t^{\alpha+N}\text{if}\ 1\geq1k$$

If we replace 1/t in place of t we have from (3.10)

$$\exp(-a/t)(1+bt)^{\alpha-1}{}_1F_1(n,\alpha;(x+a/t).(1+bt))$$

$$= \sum_{N=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha - N)} b^N \, {}_1F_1(-\alpha - 1 + N; N + 1; ab)$$

(3.12) $ {}_1F_1(n, \alpha - N; x) t^N / N! \text{ if } k \geq 1$

and from (3.11)

$$\exp(-at)(1 + b/t)^{\alpha - 1} {}_1F_1(n, \alpha; (x + at).(1 + b/t))$$

$$= \sum_{N=0}^{\infty} (a)^N \, {}_1F_1(1 - \alpha; N + 1; ab). \, {}_1F_1(n, \alpha + N; x) t^N / N! \text{ if } 1 \geq k$$

(3.13)

Application. Put a = 0, b = 1, in (3.12) we get

$$(1 + t)^{\alpha - 1} {}_1F_1(n; \alpha; x(1 + t)) = \sum_{N=0}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(\alpha - N)} \, {}_1F_1(n; \alpha - N; x) t^N / N!$$

which is derived by Panja with the help of L. Weisner's group theoretic method.

Case II. Put a = 1, b = 0 in (3, 13) we have

$$\exp(-t) {}_1F_1(n; \alpha; x + 1) \sum_{N=0}^{\infty} {}_1F_1(n; \alpha + N; x) t^N / N!$$

Which was also obtained by Panja and basu in a different form.

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