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Common Fixed Point Theorem in Cone MetricSpaces

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ABSTRACT

Plant development advancing rhizobacteria (PGPR) are a gathering of free-living microbes that colonize the rhizosphere and advantage the root development. In this exploration we have examined about the plant Growth Promotion rhizobacteria, significance, business approach of the plant development advancing rhizobacteria and the mechanism of plant development promotion.

1. Introduction

In 1922, the Polish mathematician, Stephen Banach, demonstrated a hypothesis which guarantees, under proper conditions, the presence and uniqueness of a fixed point. This outcome is called Banach's fixed point theorem. This hypothesis gives a strategy to taking care of an assortment of applied issues in scientific science and building. From that point onward, numerous creators have expanded, summed up and improved Banach's fixed point theorem in various manners and various spaces.

In 2007, Huang and Zhang[9] presented the thought of Cone metric space and depicted there union in cone metric spaces. They additionally demonstrated some fixed point hypotheses in cone metric spaces. Turkoglu and Abuloha[17] summed up certain definitions and topological ideas of cone metric spaces and demonstrated there that each cone metric space is a topological space. They likewise summed up the idea of oppositely contractive mappings and demonstrated some fixed point hypotheses in cone metric spaces. In[1], the creators demonstrated some fixed point hypotheses in cone metric spaces which summed up those in[9]. In[11], the creators characterized the semi constriction on cone metric spaces and they demonstrated some fixed point hypotheses. For later fixed point hypotheses in cone metric spaces we allude to [see2,3,8,12,13,15,16,20].

Then again, summed up constriction mappings, presented in[4], are critical in fixed point hypothesis. After that and in the most recent decade, in[7], J. Gornicki, B. E. Rhoades utilized summed up compression mappings to acquire basic fixed point hypotheses.

In this paper, we to demonstrate a fixed point hypothesis and a typical fixed point hypothesis for Ciric-type constriction planning in the setting of Cone metric spaces. Our outcomes expands and sums up the consequences of Ciric in [5] and Turkoglu et al.[18]

2. Preliminaries

Let Ebe a real Banach space and P a subset of E . Then, P is called a cone ifand only ifP1) P is closed, non-empty and$P \neq \{0\}$,

P2) $a, b \in R$; $a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$,

P3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$with respect to$P$ by $x \leq y$if andonly$y - x \in P$. We write$x < y$ to indicate that $x \leq y$ but $x \neq y$,while $x \ll y$will standfor$y - x \in IntP$. Here $IntP$denotes the interior of P .
The cone P is called normal if there is a number K , such that for all $x, y \in E, 0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$, where K is called the normal constant of P .

Thecone Piscalledregularifeveryincreasingsequencewhichisboundedfromaboveisconvergent.Thatis,if $\{x_n\}$isasequencesuchthat $x_1 \leq x_2 \leq \cdots \leq x_n \leq y$forsome
$y \in E$. Then there is $x \in E$ such that$\|x_n - x\| \to 0$ as $n \to \infty$.
Equivalently, the cone P is called regular if every decreasing sequence which is bounded below is convergent [14]. P is calledaminihedralconeifsup$\{x, y\}$existsforall$x, y \in E$,andstronglyminihedralifeverysubset of E which is bounded from above has a supremum and hence any subset of E which is bounded from below has an infimum [6]. Throughout this paper we assume that the cone P is normal with constant $K$and P is a cone in E with int$P \neq \phi$and $\leq$is a partial ordering withrespect to P .

Definition 2.1 [9]. A cone metric space is an ordered pair (X, d), where X is any set and$d : X \times X \rightarrow E$ is a mapping satisfying:

D1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
D2)$d(x,y)=d(y,x)$forall $x, y \in X$,
D3)$d(x,y) \leq d(x,z)+d(z,y)$forall$x, y, z \in X$.

Definition 2.2[9]. Let (X,d)be a cone metricspace,$\{x_n\}$a sequence in X and $x \in X$.Ifforany $c \in E$with c >>0,thereis Nsuchthatforall $n>N$, $d(x_n,x) <<$ c, then$\{x_n\}$issaid to be convergent and $\{x_n\}$converges to x. (i.e. $\lim x_n = x$ or$x_n \rightarrow x$ as $n \rightarrow \infty$).

Definition 2.3 .Let (X, d)be a cone metric space, $\{x_n\}$ asequence in$X$, if for any $c \in E$with c >>0,thereis Nsuchthatfor all $n,m>N$, $d(x_n,x_n) << c$, then$\{x_n\}$iscalled a Cauchy sequence in X.

Lemma2.1[9]. Let(X,d)beaconemetricspace,Panormalconewithanormalconstant$K$. Let $\{x_n\}$and $\{y_n\}$be twosequences in $X$and$d(x_n, y_n) \rightarrow d(x, y)$as $n \rightarrow \infty$.$y_n \rightarrow y$, $x_n \rightarrow x$ as $n \rightarrow \infty$, then

Lemma2.2[9]. Let(X,d)beaconemetricspace,Panormalconewithanormalconstant K.Let$\{x_n\}$beasequencein X.Then$\{x_n\}$convergesto $x$ifandonlyif $d(x_n,x) \rightarrow 0$as $n \rightarrow \infty$.

Lemma2.3[9]. Let(X,d)beaconemetricspace,Panormalconewithanormalconstant$K$.Let$\{x_n\}$beasequencein X.Then$\{x_n\}$Cauchysequenceifandonlyif $d(x_n,x_m) \rightarrow 0$as $m, n \rightarrow \infty$.

3. Main Results

For $x_1, x_2 \in X$ the scalar distant $d_c(x_1,x_2)$between x_1 and x_2 is defined by $d_c(x_1, x_2)=d(x_1, x_2)$.

Theorem3.1. Let(X,d)beacompleteconemetricspacewithanormalconstant $K \geq 1$ and$T : X \rightarrow X$a self-map on X such that for each $x, y \in X$

$d(Tx,Ty) \leq a(x\ )y\max d(,x\ y,)d(,x\ Tx,)d(,y\ Ty,)[d(,x\ Ty)+d(,y\ Tx)]$(3.1)

$\{d_c(xTx,)d_c(yTy)\}+(c,x),y[d_c(xTy)+d_c(yTx)]$

wherea, b, c are functions from $X \times X$into [0,1) such that

$\lambda = \sup\{Ka(x,y)b(x, y): K,cy,xy \in X\}$ (3.2)

(i) T has a unique fixed point , say $u \in X$

Proof: Fix $x \in X$.Let $\{x_n\}$be defined by From(2.1),

$d_c(x_n,x_{n+1})=d_c(Tx_{n-1},Tx_n)$ $x_0 = x$, $x_1 = Tx_0$, $x_2 = Tx_1, \cdots, x_{n+1} = Tx_n, \cdots$

$a\max\{d_c(x_{n-1},x_n),d_c(x_{n-1},Tx_{n-1}),d_c(x_n,Tx_n),d_c(x_{n-1},Tx_n)+d_c(x_n,Tx_{n-1})$

$= b\max\{d_c(x_{n-1},Tx_{n-1}),d_c(x_n,Tx_n)\}+c[d_c(x_{n-1},Tx_n)+d_c(x_n,Tx_{n-1})]$

$a\max\{d_c(x_{n-1},x_n),d_c(x_{n-1},x_n),d_c(x_n,x_{n+1}),[d_c(x_{n-1},x_{n+1})+d_c(\bar{x}_n,x_n)]$

$= b\max\{d_c(x_{n-1},x_n),d_c(x_n,x_{n+1})\}+c[d_c(x_{n-1},x_{n+1})+d_c(x_n,x_n)+ b\max\{d_c(x_{n-1},x_n),d_c(x_n,x_{n+1})\}+cd_c(x_{n-1},x_{n+1})$

wherea,b,careevaluateat(x_{n-1},x_n).

Now by Triangular inequality, wehave

$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n)+d(x_n, x_{n+1})$

Hence, $d_c(x_{n-1},x_{n+1}) \leq Kd(x_{n-1},x_n)+d(x_n,x_{n+1}) \leq K\left[d(x_{n-1},x_n)+d(x_n,x_{n+1})\right]$

$\leq 2K\max\{d(x_{n-1},x_n), d(x_n, x_{n+1})\}$

By (3.4) equation (3.3) turn to be(3.3)

$d_c(x_n,x_{n+1}) \leq (Ka+b)\max\{d_c(x_{n-1},x_n),d_c(x_n,x_{n+1})\}+2Kc\max\{d_c(x_{n-1},x_n),d_c(x_n,x_{n+1})\}$

Then, $d_c(x_n,x_{n+1}) \leq \lambda\max\{d_c(x_{n-1},x_n),d_c(x_n,x_{n+1})\}$

Since $\lambda<1$, we have$d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n)$

Proceeding in this way, we obtain $d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n) \leq \lambda^2 d(x) \leq \lambda^n d(x,Tx)$(3.6) Now we willprove that$\{x_n\}$isaCauchysequence.

For this using normality of cone, equation (3.6) and that · satisfies the triangular inequality, we obtain

$d_c(x_n,x_m) \leq K[d_c(x_n,x_{n+1})+d_c(x_{n+1},x_{n+2})+\cdots+d_c(x_{m-1},x_m)] \leq K[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)] \leq K[$

$d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+\cdots+d(x_{m-1},x_m)]$

$\leq K[\quad \lambda^n d_c(x,Tx)+\lambda^{n+1}d_c(x,Tx)+\cdots+\lambda^{m-1}d_c(x,Tx)]$

$Kd_c \quad (x,Tx)\ d_c \quad (x_n \quad , \quad x_m \quad) \leq 1-\lambda Kd_{c_n} \quad (x,Tx)$(3.7)Lettinglimitas $m,n\to\infty$in(3.7),Lemma 2.2impliesthat$\{x_n\}$isaCauchysequence.Since$(X\ ,d\ )$is a complete cone metric space, then there exists $u\ \in X$ such thatlim$x=u$

Next, we will show that u is a fixed point of T . From (3.1) and (3.2)(3.8)

$d(Tu,Tx) \leq a\ \max\ d(u,x),\ d(u,Tu),\ d(x,Tx), d(u,Tx)+d(x,Tu)$

$\leq b\max\{d_c(u,Tu),d_c(x_n,Tx_n)\}+c[d_c(u,Tx_n)+d_c(x_n,Tu)] \leq c$

$\{d_c(ux_n)d_c(uTu)d_c(x_nx_{n+1})d_c(ux_{n+1})d_c(x_nTu)\}(3.8)\leq\lambda\max\{d_c(u,x_n),d_c(u,Tu),d_c(x_n,x_{n+1}),d_c(u,x_{n+1}),d_c(x_n,Tu)\}$

Letting $n\to\infty$, then by (3.8) and Lemma 2.1, we have

$d_c(u,Tu) \leq \lambda d_c(u,Tu)$ (3.9)

Since$\lambda<1$,then$d_c(u,Tu)=0$.Hence $d(Tu,u)=0$ implies $Tu=u$.

To prove the uniqueness of fixed point, assume

T . Then from (3.1),

$d_c(x,y)=d_c(Tx,Ty)$ $x,y\in X$ and $x\neq y$ are two fixed points of

$a\max[d_c(x,y),d_c(x,Tx),d_c(y,Ty),d_c(x,Ty)+d_c(y,Tx)]\ +b\max\{d_c(x,Tx),d_c(y,Ty)\}+c\left[d_c(x,Ty)+d_c(y,Tx)\right] \leq c\ d_c\ (x$

$y) \leq \lambda d_c\ (x\ y)$

Since $\lambda<1$,then $d_c(x,y)=0$ which implies $x=y$. Since $x\in X$be arbitrary then from(3.8), we conclude that (ii) holds.

To show (iii), taking the limit in (3.7) as $n\to\infty$ and making use of Lemma 2.1, we get

$d(T^n x,u) \leq 1-\lambda d_c\ (x,Tx)$for each n . This completes the proof of the theorem.

If we put $b=c=0$ in Theorem 3.1, we get the following corollary as generalization of Theorem1 of [18] as a special case.

Corollary 3.1. Theorem 3.1. Let $(X\ ,d\ )$be a complete cone metric space with a normalconstant $K\geq 1$ and $T:X\to X$a self-map on X such that for each $x,y\in X$:

$d(Tx,Ty) \leq \lambda\max d(x,y),d(x,Tx),d(y,Ty),\ d(x,Ty)+d(y,Tx)$

where$\lambda\in(0,1)$ with $\lambda K<1$, then

(i) T has a unique fixed point , say $u\in X$,

(ii)$T^n x\to u$as $n\to\infty$, for each $x\in X$,

$(Tx,u) \leq 1-\lambda d_c\ (x,Tx)$

Let Sbeanon-emptysetandlet$\{T_\alpha\}_{\alpha\in J}$beafamilyofself-mappingson $S$and J anindexingset.Apoint$u\in S$iscalledacommonfixedpointforafamily$\{T_\alpha\}_{\alpha\in J}$ifandonly if $u=T_\alpha u$ for each T_α.

Theorem3.2.Let(X,d)beacompleteconemetricspacewithanormalconstant $K\geq 1$ and

$\{T_\alpha\}_{\alpha\in J}$afamilyof self-mappings of X. If there exists a fixed$\alpha\in J:\beta\in J$ such that for each$d(T\,x,Ty) \leq a\ \max d(x,y),d$ $(x,Tx),d(y,Ty),d(x,Ty)+d(y,Tx)\ +b\max\{d_c(x,T_\alpha x),d_c(y,T_\beta y)\}+c\left[d_c(x,T_\beta y)+d_c(y,T_\alpha x)\right]$

where$a$, b, c are functions from $X\times X$into [0,1) such that

Thenall$T_\alpha$haveauniquecommonfixedpointwhichisauniquefixedpointofeachT_α,$\alpha\in J$.

Proof: Let$\alpha\in J$ and $x\in X$be arbitrary. Consider a sequence defined by$x_0=x$,

$x_{2n+1}=T_\alpha x_{2n}$,$x_{2n+2}=T_\beta x_{2n+1}$, $n\geq 0$. From (3.10), we get

$d_c(x_{2n+1},x_{2n+2})=d_c(T_\alpha x_{2n},T_\beta x_{2n+1})\ a\max d_c(x_{2n},x_{2n+1}),d_c(x_{2n},T_\alpha x_{2n}),d_c(x_{2n+1},T_\beta x_{2n+1}),[d_c(x_{2n},T_\beta x_{2n+1})+$ $d_c(x_{2n+1},T_\alpha x_{2n})]\ +b\max\{d_c(x_{2n},T_\alpha x_{2n}),d_c(x_{2n+1},T_\beta x_{2n+1})\}+cd_c(x_{2n},T_\beta x_{2n+1})+d_c(x_{2n+1},T_\alpha x_{2n}]\ a\ \max\ \{d_c$ $(x_{2n}\ ,\ x_{2n+1}\ ),\ d_c\ (x_{2n}\ ,\ x_{2n+1}\ ),\ [d_c\ (x_{2n+1}\ ,\ x_{2n+2}\ ),\ d_c\ (x_{2n}\ ,\ x_{2n+2}\ )+d_c\ (x_{2n+1}\ ,\ x_{2n+1}$ $)]\ +b\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}+cd_c(x_{2n},x_{2n+2})+d_c(x_{2n+1},x_{2n+1})$

Since

$d(x_{2n},x_{2n+2}) \leq d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})\ d(x_{2n}x_{2n+2}) \leq Kd(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2})$ (3.12)

$d_c(x_{2n},x_{2n+2}) \leq K[d_c(x_{2n},x_{2n+1})+d_c(x_{2n+1},x_{2n+2})]$

So,

$d(x,x)\leq K\, d(x,x)+d(x,x)\leq K\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}$

We have, $d_c(x_{2n+1},x_{2n+2})\leq aK\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}+b\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}+cK\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}$

Now,

Case I.If $\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}=d_c(x_{2n},x_{2n+1})$, then $\leq\lambda d_c(x_{2n},x_{2n+1})$

Case II.If $\max\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\}=d_c(x_{2n+1},x_{2n+2})$, then $\leq\lambda d_c(x_{2n},x_{2n+1})$ Similarly,wemayfindthat $d_c(x_{2n},x_{2n+1})\leq\lambda d_c(x_{2n-1},x_{2n})$. Thus for any $n\geq 1$, wehave $d(x,x)\leq\lambda d(x,x)\leq\lambda^2 d(x,x)\leq\cdots\leq\lambda^n d(x,x)$.

Now, we show that $\{x_n\}$is a Cauchy sequence in X. For $m>n$, we get

$$d_c(x_n,x_m)\leq K\left[d_c(x_n,x_{n+1})+d_c(x_{n+1},x_{n+2})+\cdots+d_c(x_{m-1},x_m)\right]\leq K\left[\lambda^n d(x,x)+\lambda^{n+1}d(x,x)+\cdots+\lambda^{m-1}d(x_0,x_1)\right]$$
$$\leq K\left[\lambda^n+\lambda^{n+1}+\cdots+\lambda^{m-1}\right]d(x,x) \quad (3.13)$$

Lettinglimitas $m,n\to\infty$ andusingLemma2.2,weconcludethat $\{x_n\}$ isaCauchysequence. Since X is complete, there is a $z\in X$ such that

From (3.10), we have

$d_c(T_\beta z,x_{2n+1})=d_c(T_\beta z,T_\alpha x_{2n})\ \lim x=z$ (3.14)

$a\max\{d_c(z,x_{2n}),d_c(z,T_\beta z),d_c(x_{2n},T_\alpha x_{2n}),[d_c(z,T_\alpha x_{2n})+d_c(x_{2n},T_\beta z)]$
$+b\max\{d_c(z,T_\beta z),d_c(x_{2n},T_\alpha x_{2n})\}+cd_c(z,T_\alpha x_{2n})+d_c(x_{2n},T_\beta z)$

Taking the limit as $n\to\infty$ and using Lemma 2.1, we get $d_c(T_\beta z,z)\leq(a+b+c)d_c(T_\beta z,z)\Rightarrow d_c(T_\beta z,z)=0$ as $a+b+2c<1$ and so $T_\beta z=z$. Nowweshowthat zisa fixedpointof all $\{T_\alpha\}_{\alpha\in J}$,let $\alpha\in J$ bearbitrary.Thenfrom(3.10) with $x=y=z=T_\beta z$, wehave

$d_c(z,T_\alpha z)=d_c(T_\beta z,T_\alpha z)=a\max\{d_c(z,z),d_c(z,T_\beta z),d_c(z,T_\alpha z),d_c(z,T_\beta z)+d_c(z,T_\beta z)+b\max\{d_c(z,T_\beta z),d_c(z,T_\alpha z)\}+cd_c(z,T_\beta z)+d_c(z,T_\beta z)$

$\leq(a+b+c)d_c(z,T_\alpha z)$

and hence $T_\alpha z=z$. Thus all T_α have a common fixed point.

Suppose that w is another fixed point of T_β. Then as proved above, w is a commonfixed point of all $\{T_\alpha\}_{\alpha\in J}$. Thus from (3.10), we have

$d_c(z,w)=d_c(T_\beta z,T_\alpha w)\leq(a+b+c)d_c(zw)$ and so $z=w$.Thuszis a unique common fixed pointof all $\{T_\alpha\}_{\alpha\in J}$.

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