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Definition and Algebra of Fuzzy Sets

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ABSTRACT

In this paper our aim is to discuss Fuzzy sets, Fuzzy Null set, subset of Fuzzy sets, equality of two Fuzzy sets, compliment of Fuzzy set, union and Intersection of two Fuzzy sets, Algebraic Product, Algebraic sum, Bold Union, Bold Intersection, Bounded Difference, symmetric difference, power of Fuzzy set, convex and concave Fuzzy set, Fuzzy number Cartesian product of Fuzzy set, Extension principle and other important definition.

DEFINITION

The introduction of the concept of fuzzy set and the formalization of the basic in operations in language of fuzzy sets leads in natural way to the idea of the generalization of the classical set theory for fuzzy set. Fuzzy set theory has a vast range. It has application not only in the branch of mathematics but also in social sciences, medicine, mines, industry etc. It has also applications in different branches of mathematics such as topology, linear space, group, statistics etc. After 1965 it is increasing continuously and its applications got a vast range. The way of looking our problems today differ from the problem of the past. We are no longer merely facing the world trying to win its secrets or battling to shape it to meet our desire. We are now facing our selves and our knowledge and trying to cope with the complexity, we have created. Hence we no longer formulate the problem in our scientific enquiries or in our engineering attempts simply in terms of missing facts or inadequate measuring devices or materials. Problems are seen to be of a different order, that is , they concern question about knowledge and information itself. Therefore, it can be formulated independently of any particular area of knowledge. The application of fuzzy set theory include explanations within psychology and cognitive science of concept formation and manipulation memory and learning, as well as studies in field of sociology, Economics, Ecology, Meteorology, Biology and others.

Most probability the idea of fuzzy set came into the mind of L.A. Zadeh from the definition of characteristic function. The process by which individuals from the universal set X are determined to be either member or non members of a set can be defined by a characteristic functions. For a given set A, this function assigns a value $\mu A(x)$ to every $x \in X$ such that

$$\mu_A(x) = \begin{array}{l} 1 \text{ if and only if } x \in A \\ 0 \text{ if and only if } x \notin X \end{array}$$

Thus, the function maps elements of the universal set to the set containing 0 and 1. This can be indicated by

$$\mu A \ : X \longrightarrow \{ 0 , 1\}$$

But we see in the real world there are so many sets which have no clear boundaries and cannot be defined by characteristic function. For example if we consider the set of all tall men, the set of beautiful persons, the set of talented people etc. the word tall men, beautiful persons, talented people used in the above set have no clear boundaries and it cannot be defined by characteristic function. It can be only defined in terms of membership grade. Thus we see that the characteristic function can be generalized such that the valued assigned to the elements of the universal set of all contain a specified range and indicate the membership grade of those elements in the set in question. Larger values denotes higher degrees of set membership. Such a function is called a membership function and the set defined by it a fuzzy set.

1.1 FUZZY SET:-

Let X denotes a universal set. Then the membership function $\overline{A}$ (x) by which a fuzzy set A is usually defined has the form

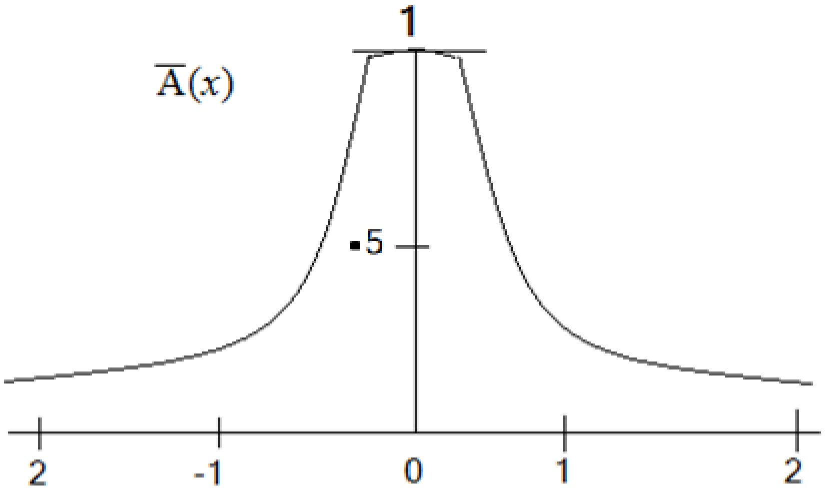
$$\overline{A}(x) : X \longrightarrow \{0, 1\}$$

Where [0, 1] denotes the interval of real numbers from 0 to 1, inclusive.

For example, we can define a possible membership function for the fuzzy set of real number close to O as follows:

$$\overline{A}(x) = \frac{1}{1 + 20x}$$

The graph of the function is pictured below.



Using this function, we can determine the membership grade of each real number in this fuzzy set, which signifies the degree to which that number is close to 0. For instance the number 3 is assigned a grade of .01, the number 2 a grade of .02, the number 1 a grade of .04, the number .25 a grade of .4 and the number 0 a grade of 1. We might intuitively expect that by performing some operation on the function corresponding to the set of number close to 0.

1.2 FUZZY NULL SET:-

Fuzzy null set φ is given by the gradation function.

$$\varphi(x) = 0 \quad x \in X$$

The universal set X is given by the gradation function X(x) =1 $\forall x \notin X$

Obviously, the fuzzy null set φ and the universe X are ordinary set.

1.3 SUBSET OF FUZZY SETS:-

If the membership grade of each element of the universal set X in fuzzy set $\overline{A}$ is less than or equal its membership grade in fuzzy set $\overline{B}$, then $\overline{A}$ is called subset of $\overline{B}$. Hence if

$$\overline{A}(x) \leq \overline{B}(x) \ \forall x \in X$$

Then $\overline{A} \subseteq \overline{B}$.

1.4 EQUALITY OF TWO FUZZY SETS:-

Fuzzy sets $\overline{A}$ and $\overline{B}$ are equal if $\overline{A}(x) = \overline{B}(x)$ for every element $x \in X$. It is denoted by $\overline{A} = \overline{B}$. If fuzzy sets $\overline{A}$ and $\overline{B}$ are not equal if $\overline{A}(x) \neq \overline{B}(x)$ for at least one $x \in X$. We write $\overline{A} \neq \overline{B}$.

1.5 PROPER FUZZY SUBSET:-

Fuzzy set $\overline{A}$ is called a proper subset of fuzzy set $\overline{B}$ when $\overline{A}$ is a subset of $\overline{B}$ and the two sets are not equal.

That is if $\overline{A}(x) \leq \overline{B}(x) \; \forall \, x \in X$ and $\overline{A}(x) < \overline{B}(x)$ for at least one $x \in X$. We denote proper subset by $\overline{A} \subset \overline{B}$ if $\overline{A} \subseteq \overline{B}$ and $\overline{A} \neq \overline{B}$.

1.6 COMPLIMENT OF $\overline{A}$ FUZZY SET:-

Let $\overline{A}'$ denote the compliment of fuzzy set $\overline{A}$, then we define

$$\overline{A}'(x) = 1 - \overline{A}(x) \; \forall \, x \in X$$

Now we set an example to understood the above definitions.

Let X = { 5,10,20,30,40,50,60,70,80} be an universal set of ages.

Let us consider the fuzzy sets infant, adult, young, old are four of the elements of the power set containing all possible fuzzy subset of X.

Now we arrange the above facts by the table given below:

Elements (ages)	Infant	Adult	Young	Old
5	0	0	1	0
10	0	0	1	0
20	0	.8	.8	.1
30	0	1	.5	.2
40	0	1	.2	.4
50	0	1	.1	.6
60	0	1	0	.8
70	0	1	0	1
80	0	1	0	1

The fuzzy set old from table is a subset of the fuzzy set adult, since for each element in our universal set

$$\overline{A}_{old}(x) \leq \overline{A}_{adult}(x)$$

None of the four fuzzy set defined in the table is equal to any of the others. That is, these sets in the table are unequal. Since fuzzy set old from table is a subset of the fuzzy set adult and that these two fuzzy sets are not equal. Thus we see that fuzzy set old is a proper subset of fuzzy set adult. If we take an element which has a membership grade of .8 in a fuzzy set $\overline{A}$, its membership grade in complement of $\overline{A}$ will be .2. It is obvious that the fuzzy set of infant is an example of null fuzzy set.

1.7 UNION OF TWO FUZZY SETS:-

The Union of two fuzzy sets $\overline{A}$ and $\overline{B}$ is the least fuzzy set on X which contains both fuzzy sets $\overline{A}$ and $\overline{B}$ and is denoted by $\overline{A} \cup \overline{B}$.

The membership function of the fuzzy set $\overline{A} \cup \overline{B}$ is defined by

$$\overline{A} \cup \overline{B}(x) = \max\{\overline{A}(x), \overline{B}(x)\} \; \forall \, x \in X$$

In other words

$$\overline{A} \cup \overline{B}(x) = \overline{A}(x) \vee \overline{B}(x) \; \forall \, x \in X.$$

More in generalized way, for a family of fuzzy sets

$$\overline{A} = \{\overline{A}_i\}_{i \in I}$$

The union C = $\bigcup_{i \in I} \overline{A}_i$ is defined by

$$C(x) = \sup_{I} \{\overline{A}_i(x)\},\ x \in X$$

1.8 INTERSECTION OF TWO FUZZY SETS:-

The intersection of two fuzzy sets $\overline{A}$ and $\overline{B}$ is the greatest fuzzy set on X which is a subset of both these sets and is denoted by $\overline{A} \cap \overline{B}$.

The membership function of the fuzzy set $\overline{A} \cap \overline{B}$ is defined by

$$\overline{A} \cap \overline{B}(x) = \min \{\, \overline{A}\,(x), \overline{B}(x)\} \ \forall x \in X$$

In other words we can say

$$\overline{A} \cap \overline{B} = \{\overline{A}(x) \wedge \overline{B}\} \ \forall x \in X$$

In general way for the family of fuzzy sets

$\overline{A} = \cap \{\overline{A}_i\}_{i \in I}$ the intersection

D = $\bigcap_{I} \overline{A}_i$ is defined by gradation function

$$D(x) = \inf_{I} \{\overline{A}_i(x)\},\ x \in X.$$

1.9 ALEGBRAIC PRODUCT:-

The algebraic product of two fuzzy sets A and B denoted by $\overline{A} \cdot \overline{B}$ is defined in terms of membership function as

$$\overline{A} \cdot \overline{B}(x) = \overline{A}(x) \cdot \overline{B}(x) \ \forall x \in X$$

1.10 ALGEBRAIC SUM:-

The algebraic sum of two fuzzy sets A and B denoted by $\overline{A} \oplus \overline{B}$ is defined by membership function as

$$\overline{A} \oplus \overline{B}\,(x) = \overline{A}(x) + \overline{B}(x) - \overline{A}(x) \cdot \overline{B}(x) \ \forall x \in X$$

1.11 BOLD UNION:-

The bold union of two fuzzy sets $\overline{A}$ and $\overline{B}$ on x denoted by $\overline{A} \Cup \overline{B}$, is defined by membership function as $\overline{A} \Cup \overline{B}(x) = \min\,[1, \overline{A}(x) + \overline{B}(x)] \ \forall x \in X$.

1.12 BOLD INTERSECTION:-

The bold intersection of two fuzzy sets $\overline{A}$ and $\overline{B}$, denoted by $\overline{A} \quad \overline{B}$, is defined by membership function as

$$\overline{A} \quad \overline{B}\,(x) = \max\,(0, \overline{A}(x) + \overline{B}(x) - 1) \ \forall x \in X$$

1.13 BOUNDED DIFFERENCE:-

The bounded difference of two fuzzy set on X, denoted by $\overline{A}| - |\overline{B}$ is defined by membership function as $\overline{A}| - |\overline{B}(x) = \max\,(0, \overline{A}(x) - \overline{B}(x)) \ \forall x \in X$

i.e., $\overline{A}| - |\overline{B}$ is the fuzzy sets of elements that belong to $\overline{A}$ more than to $\overline{B}$.

1.14 SYMMETRIC DIFFERENCE:-

The symmetric difference of two fuzzy sets $\overline{A}$ and $\overline{B}$, denoted by $\overline{A} \nabla \overline{B}$ is defined by membership function as

$$\overline{A} \nabla \overline{B}(x) = |\overline{A}(x) - \overline{B}(x)| \ \forall x \in X$$

i.e., $\overline{A} \nabla \overline{B}$ is a fuzzy set of elements that belong more to $\overline{A}$ then to $\overline{B}$ or conversed sometimes the systematic difference of two fuzzy sets $\overline{A}$ and $\overline{B}$ is denoted by $\overline{A} \Delta \overline{B}$ and is defined by membership function as

$$\overline{A} \Delta \overline{B}(x) = \max \{\min (\overline{A}(x), 1-\overline{B}(x)), \ \min (1-\overline{A}(x), \overline{B}(x))\} \ \forall x \in X$$

i.e., the fuzzy set $\overline{A} \Delta \overline{B}$ is the elements that approximately belong to $\overline{A}$ and not to $\overline{B}$ or conversely to $\overline{B}$ and not to $\overline{A}$.

1.15 POWER OF FUZZY SET:-

The mth power of a fuzzy set $\overline{A}$, denoted by $\overline{A}^m$ is defined by membership function as

$$\overline{A}^m (x) = [\overline{A}(x)]^m \ \forall x \in X \text{ and } \forall m \in R^+$$

1.16 L-FUZZY SET:-

Let L be an arbitrary set, then L - fuzzy set $\overline{A}$ is defined by a membership function

$$\overline{A}(x) : X \to L$$

Where L may have of different algebraic structures such as group, semigroup, ring, semi ring and lattice.

If we consider L to be a lattice then union and intersection of L – fuzzy sets is defined as

$$\overline{A} \cap \overline{B}(x) = \inf (\overline{A}(x), \overline{B}(x)) \ \forall x \in X$$

$$\overline{A} \cup \overline{B}(x) = \sup (\overline{A}(x), \overline{B}(x)) \ \forall x \in X$$

1.17 CONVEX FUZZY SET:-

The fuzzy set $\overline{A}$ on real Eucledean N – dimensional space X is said to be convex if for any pair of elements x,y in X the membership function $\overline{A}(x)$ satisfies the condition.

$$\overline{A}[\lambda x + (1 - \lambda)y] \geq \min (A(x), A(y)), \ 0 \leq \lambda \leq 1$$

1.18 CONCAVE FUZZY SET:-

A fuzzy set $\overline{A}$ is said to be concave if $\overline{A}$ is complement of a convex set.

if

$$\overline{A}[\lambda x + (1 - \lambda)y] \leq \max (A(x), A(y)) \ \forall x \in X \text{ and } 0 \leq \lambda \leq 1$$

1.19 FUZZY NUMBER:-

A fuzzy number is a convex normalized fuzzy set $\overline{A}$ on the real line R such that

(a) there exists $x_0 \in R$, for which

$\overline{A}(x_0) = 1$ (x_0 is called the mean value of A)

(b) $\overline{A}(x)$ is piecewise continuous.

1.20 POSITIVE AND NEGATIVE FUZZY NUMBERS:-

A fuzzy number on R is said to be positive if $\overline{A}(x) = 0 \ \forall x < 0$

And that of negative if

$\overline{A}(x) = 0 \ \forall x > 0$

1.21 MEASURE OF FUZZINESS OF A FUZZY SET:-

If we denote the set of all fuzzy sets on X by $\tilde{p}(x)$ then a measure of fuzziness is a mapping d from $\tilde{p}(x)$ to $[0, +\infty]$ satisfying the following conditions

(1) d(A) = 0 iff A is an ordinary subset of X.

(2) d(A) is maximum iff $A(x) = \frac{1}{2} \forall x \in X$

(3) $d(A^*) \leq d(A)$, where A* is any sharpend version of A, that is

$A^*(x) \leq A(x)$ if $A(x) \leq \frac{1}{2}$

And $A^*(x) \geq A(x)$ if $A(x) \geq \frac{1}{2}$

(4) d(A) = d(A*) (A) is as fuzzy as A.).

1.22 CARTESIAN PRODUCT OF FUZZY SETS:-

Let $X_1, X_2, X_3, \ldots\ldots, X_r$ are the universal sets and X be their cartesian product

i.e., $X = X_1 \text{ x } X_2 \text{ x } X_3 \text{ x } \ldots\ldots \text{x } X_r$

If $A_1, A_2, A_3, \ldots\ldots, A_r$ be fuzzy sets on $X_1, X_2, X_3, \ldots\ldots, X_r$ respectively than the Cartesian product of fuzzy sets $A_1, A_2, \ldots\ldots, A_r$ is defined by membership function as

$A_1 x A_2 x A_3 x \ldots\ldots x A_r(X_1, X_2, X_3, \ldots\ldots X_r) = \min(A_1(X_1), A_2(X_2), \ldots\ldots, A_r(X_r))$

Where $X_1 \in X_1, X_2 \in X_2, \ldots\ldots$ and $X_r \in X_r$

1.23 ∝ - CUT OF A FUZZY SET:-

An ∝ - cut of a fuzzy set is an ordinary set, denoted by $A_{\propto}$, that contains all the elements of the universal set X that have a membership grade in a greater than or equal to the specified of ∝. This definition can be written as

$A_{\propto} = \{ x \in X / A(x) \geq \propto \}$

The value can be chosen arbitrarily but is often designated at the value of the membership grades appearing in the fuzzy set under consideration.

EXTENSION PRINCIPLE

L.A. Zadeh introduced the extension principle * on fuzzy set, which provides a general method for extending non – fuzzy mathematical concept to corresponding fuzzy concepts.

Let $x = \{ x_i \}$ and $Y = \{ x_j \}$, where x_i and y_j are generic element of X and Y respectively.

Let $f : x \rightarrow y$ (f is one - one) and A be any fuzzy set on X given by

$\mu(x_i) = a_i$ (say)

Let $f(x_i) = y_i$, we can extend the mapping f to fuzzy sets on x as follows:

Let A be any fuzzy set on x then f(A) is taken to be a fuzzy sets on y given by

$f(A)(Y_i) = a_i$

Thus the image of under f can be known from the knowledge of images of elements of x under f.

Thus domain of f is extended from x to fuzzy sets on x.

Similarly we determine fuzzy set on a universe Y from given fuzzy sets $A_1, A_2, A_3, \ldots\ldots\ldots, A_r$ on universe $X_1, X_2, \ldots\ldots\ldots X_r$ respectively through a mapping of from the Cartesian product $X_1 \times X_2 \times X_3 \times \ldots\ldots \times X_r$ to the given universe Y.

❖ L.A. Zadeh, The concept of linguistic variable and its application to approximate reasoning. Inform, Sci. 8(1975), 199-249, 301 – 357, 9(1975), 43 – 80.

1.24 DEFINITION:-

Let $f : X_1 \times X_2 \times X_3 \times \ldots\ldots \times X_r \rightarrow Y$ is a mapping such that

$y = f(x_1, x_2, \ldots\ldots, x_r)$ where $x_1 \in X_1$, $x_2 \in X_2$ ……….and $x_r \in X_r$.

Let $A_1, A_2, \ldots\ldots, A_r$ be the fuzzy sets on universes $X_1, X_2, \ldots\ldots X_r$ respectively then for given fuzzy sets $A_1, A_2, \ldots\ldots, A_r$ we define a fuzzy sets B in Y through the mapping f as

$$B(y) = \text{Sup. Min}\{A_1(x_1), A_2(x_2), \ldots\ldots A_r(X_r)\}\ x_1, x_2, \ldots\ldots, x_r$$

$$y = f(x_1, x_2, \ldots\ldots, x_r)$$

$$= 0 \text{ if } f^{-1}(y) = \varphi$$

Where $f^{-1}(y)$ is the inverse image of y i.e., B(y) is the greatest among the membership value $(A_1 \times A_2 \times A_3 \times \ldots\ldots \times A_r)(x_1, x_2, x_3, \ldots\ldots, x_r)$ of the realization of y through f using r – tuples (x1, x2,…….., xr). The fuzzy set B is sometime denoted by $f(A_1, A_2, \ldots\ldots, A_r)$.

If $f : x_1 \rightarrow Y$ is a mapping and A is a fuzzy set in x_1, then we define a fuzzy set B in Y as

$$B(y) = \underset{x_1}{\text{Sup}}.\ \min\{A_1(x_1)\}$$

$$y = f(x_1)$$

$$= \underset{x_1}{\text{Sup}}.\ \min\{A_1(x_1)\}$$

$$y = f(x_1)$$

and $B(y) = 0$ if $f^{-1}(y) = \varphi$.

If f is one – one $B(y) = A_1(f^{-1}(y)) = 0$ if $f^{-1}(y) = \varphi$

REFERENCES

1. L.A. Zadeh, Fuzzy sets, Information and control, 8(1965)338-353.
2. Kandel & Byatt, W.J (1978) – Fuzzy sets, Fuzzy Algebra and Fuzzy Statistics, Proc-IEEE 66, 1619.
3. LEER.C.T and CHANGC.L, (1971), some properties of Fuzzy Logic, Information and Control, 19, 417.
4. Kandel A, and Byatt, W.J. Fuzzy Sets, Fuzzy algebra and fuzzy statistics Proc-IEEE 66 1619 – 1637.
5. Kaufmann, A (1975) Introduction to the theory of fuzzy subsets, Vol.I New York, Sanfrancisco, London.