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On Stability of Weighted Frames in Banach Spaces

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ABSTRACT

In this paper we study the Paley-Weiner type stability theorem for weighted X_d- frames in Banach space.

1. Introduction

Duffin and Schaeffer [7], introduced frames for Hilbert spaces in 1952. Lateron, in 1986, Daubechies, Grossmann and Meyer [8] found a fundamental new application to wavelet and Gabor's transforms in which frames play an impor- tant role. In fact, the theory of frames is a central tool in many areas such as signal processing, image processing, data compression etc. Coifman and Weiss [6] introduced the notion of *atomic decomposition* for function spaces. Later,Feichtinger and Grochening[9] extended the notion of atomic decomposition tocertain Banach spaces. Grochening[11] introduced a more general concept for Banach spaces called a *Banach frame*. Banach frames were further studied in[2, 3, 4, 11, 13].Fourier transform has been a major tool in analysis for over a century. It has a lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was localized time frequency representation which has this information encoded in it. In 1946 Dennis Gabor [10], filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952 the notion of frame was determined by Duffin and Schaeffer [7] in Hilbert spaces in the

following way: Let X be a Separable Hilbert space, the system of non-zero elements $\{ x_n \}_n \subset X$ be called a frame in X if there exist the constants $0 < A \leq B < \infty$ such that for each $x\ X$, it is valid

$$A\ ||x||^2 \leq \sum_{n=1}^{\infty} |(x, x_n)|^2 \leq B||x||^2 \qquad (1.1)$$

The constants A and B in (1.1) are called lower and upper frame bounds, the number $K=A / B$ is called condition coefficient of the frame $\{ x_n \}_n$. In the

case, when $K= 1$, $\{ x_n \}_n$ is a tight frame. Development in theory of frames in Hilbert space reduced to obtaining the analogues of the known results for the Banach case. By the theory of frames [3, 6, 7, 9] we have their Banach extensions in [1, 2, 4, 5, 10]. Frames have many properties of bases but lacks a very important one, namely, uniqueness. This property of frames make them very useful in the study of function spaces, signal and image processing, filter banks, wireless and communications etc.

In the present paper, we further study weighted Xd -frames and obtain a necessary and sufficient condition. Also, a sufficient condition for stability of X_d -frame has been given and proved if a Banach spaces has a X_d -frame then it is also has a Parseval and an exact Parseval X_d -frames.

Further, X_d -Bessel sequence has been studied and a sufficient condition in terms of X_d -Bessel sequence, for a sequence to be a X_d -frame has been given.

Definition: 1.1 ([1]) Let X_d be a BK-space and $\{w_i f_i\}_{i=1}^{\infty} \subset X^*$. The sequence $\{w_i f_i\}_{i=1}^{\infty}$ is called a weighted X_d-frame for X with lower bound A and upper bound B if $0 < A \leq B < \infty$ and for every $x \in X$ one has

(i) $\{ w_i f_i (x) \}_{i=1}^{\infty} \in X_d$;

(ii) $A\|x\|\leq\|\{ w_i f_i (x)\}_{i=1}^{\infty}\|\leq B\|x\|$ (1.2)

When (i) and the upper inequality in (ii) hold for every $x\in X$, $\{ f_i \}_{i=1}^{\infty}$ is called a X_d-Bessel sequence for X with bound A and B . The positive constants A and B, respectively, are called lower and upper frame bounds of the weighted X_d - frame.

The inequality (1.2) is called the frame inequality. It is easy to observe that frame bounds need not be unique. Further, the X_d - frame is called tight frame if it is possible to choose A and B, satisfying (2.1) with $A=B$ and normalized tight if A=B=1. If removal of one w_nf_n renders the collection $\{ w_nf_n \}$ no longer a weighted X_d frame for X, then $\{ w_nf_n \}$ is called an exact weighted X_d frames. The operator U and T given by

$U(x)= \{ w_if_i \}$ and $T(d_i)=\sum d_iw_i f_i$ are called the analysis operator for $\{ w_if_i \}_{i=1}^{\infty}$ and the synthesis operator for $\{ w_if_i \}_{i=1}^{\infty}$, respectively.

Stability theorems for frames in Hilbert spaces were studied in [1, 3, 8, 12] and for Banach frames were studied by Christensen and Heil [4].

In the present paper, we prove some stability theorems (Paley-Wiener type) for Banach frames in Banach spaces.

2. MAIN RESULTS

In the following theorem, we give a sufficient condition for a sequence to be a weighted X_d -frame for X.

Theorem 2.1 If $\{w_nf_n\}\subset X^*$ be a weighted X_d-frames for X with frame bounds A and B. Let $\{u_ng_n\}\subset X^*$, such that $\{u_ng_n (x)\} \subset X_d$, for all $x\in X$. If there exists non-negative constants α,β,γ and δ and such that

(i) $$\frac{\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}} < A, \text{ where } \sqrt{\max\{\alpha,\beta,\gamma,\delta\}} <1.$$

(ii) $$\|\{(w_nf_n-u_ng_n)(x)\}\|^2 \leq \alpha\|\{w_nf_n(x)\}\|^2 +2\beta\|\{w_nf_n(x)\}\|\|\{u_ng_n(x)\}\|$$

$$+ \gamma\|\{u_ng_n(x)\}\|^2 + \delta \|x\|^2,$$

Then $\{u_ng_n\}$ is a weighted X_d- frame for X with frame bounds

$$\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}} \text{ and } \frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}.$$

Proof. Let $\zeta = \max \{\alpha,\beta,\gamma,\delta\}$. Then (ii) may be reproduced as:

$$\|\{(w_nf_n-u_ng_n)(x)\}\| \leq \sqrt{\zeta} \{ \|\{w_nf_n(x)\}\| +\|\{u_ng_n(x)\}\| + \|x\|\}, \quad (2.1)$$

for all $x\in X$.

Now,

$$\|\{u_ng_n(x)\}\|\leq \|\{w_nf_n(x)\}\| + \|\{(w_nf_n-u_ng_n)(x)\}\|$$

$$\leq \|\{w_nf_n(x)\}\| + \sqrt{\zeta} \{ \|\{w_nf_n(x)\}\| +\|\{u_ng_n(x)\}\| + \|x\|\} .$$

by using (2.1)

This gives,

$$(1-\sqrt{\zeta})\|\{u_n g_n(x)\}\| \le \{(1+\sqrt{\zeta})B + \sqrt{\zeta}\}\|x\|, \text{ for all } x \in X.$$

Similarly,

$$(1+\sqrt{\zeta})\|\{u_n g_n(x)\}\| \ge \{(1-\sqrt{\zeta})A + \sqrt{\zeta}\}\|x\|, \text{ for all } x \in X.$$

Therefore,

$$\left\{\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}\right\}\|x\| \le \|\{u_n g_n(x)\}\| \le \left\{\frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}\right\}\|x\|.$$

for all $x \in X$.

Hence, $\{u_n g_n(x)\}$ is a weighted X_d- frame for X with frame bounds

$$\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}} \text{ and } \frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}.$$

This completes the proof.

Theorem 2.2 If $\{w_n f_n\} \subset X^*$ be a weighted X_d-frames for X with frame bounds A and B. Let $\{u_n g_n\} \subset X^*$ be weighted X_d– Bessel sequence for X with bound K<A . Then $\{w_n f_n \pm u_n g_n\}$ is a weighted X_d- frame for X.

Proof. Suppose that R_T, R_Q are the analysis operators of weighted X_d- Bessel sequences $\{w_n f_n\}$ and $\{u_n g_n\}$ for X, respectively.

For any $x \in X$, we have

$$\begin{aligned}\|\{(w_n f_n \pm u_n g_n)(x)\}\| &= \|R_T(x) \pm R_Q(x)\| \\ &\le \|R_T(x)\| + \|R_Q(x)\| \\ &\le (B+K)\|x\|, \ x \in X.\end{aligned}$$

Thus, $\{w_n f_n \pm u_n g_n\}$ is a weighted X_d- Bessel sequence for X with bound (B + K).

Also,

$$\begin{aligned}\|\{(w_n f_n \pm u_n g_n)(x)\}\| &= \|R_T(x) \pm R_Q(x)\| \\ &\ge \|R_T(x)\| - \|R_Q(x)\| \\ &\ge (A-K)\|x\|, \ x \in X.\end{aligned}$$

Hence, $\{w_n f_n \pm u_n g_n\}$ is a weighted X_d- frame for X with bound (A-K), (B + K).

Theorem 2.3 If $\{w_n f_n\} \subset X^*$ be a weighted X_d-frames for X with frame bounds A and B. Let $\{u_n g_n\} \subset X^*$ such that $\{u_n g_n(x)\} \subset X_d$. Let $\{w_n f_n + u_n g_n\}$ is a weighted X_d- Bessel sequence for X with bound K<A. Then $\{u_n g_n\}$ is a weighted X_d- frame for X with frame bounds A - K and B + K.

Proof. By using hypothesis, we have

$$(A - K) \|x\| \leq \|\{w_n f_n(x)\}\| - \|\{w_n f_n + u_n g_n(x)\}\|$$

$$\leq \|\{u_n g_n(x)\}\|$$

$$\leq \|\{w_n f_n(x)\}\| + \|\{(w_n f_n + u_n g_n)(x)\}\|$$

$$\leq (B + K) \|x\|, \quad \text{for all } x \in X.$$

Hence, $\{u_n g_n\}$ is a weighted X_d - frame for X with frame bounds A - K and B + K.

This completes the proof.

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