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An Eigen value Problem Associated with a Partial Differential Operator

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ABSTRACT

In this paper a matrix differential operator containing partial derivatives has been considered and eigen value problem associated with it has been defined. Boundary conditions and bilinear concomitant have been obtained. Existence theorem has been proved.

1. Definition of the operator

Consider the equation

$\frac{\partial^2}{\partial x^2}\Psi(x,y) - A(x,y)\,\Psi(x,y) = \lambda\,\Psi(x,y)$, $a \le x \le b$, and $c \le y \le d$; …………………. (1.1)

Where

$A(x,y) = [A_{ij}]$ is a symmetric matrix of order n; A_{ij} are real valued functions of x,y;

$\Psi(x,y) = [\Psi_1(x,y), \Psi_2(x,y), \ldots\ldots\ldots \Psi_n(x,y)]^T$ is a vector having n components; and λ is a parameter, real or complex. Equation (1.1) is equivalent to a set of differential equations given by

$$\frac{\partial^2}{\partial x^2}\Psi_1 - (A_{11}\,\Psi_1 + A_{12}\,\Psi_2 \ldots\ldots\ldots\ldots A_{1n}\,\Psi_n) = \lambda\,\Psi_1$$

$$\frac{\partial^2}{\partial x^2}\Psi_2 - (A_{11}\,\Psi_1 + A_{12}\,\Psi_2 \ldots\ldots\ldots\ldots A_{1n}\,\Psi_n) = \lambda\,\Psi_2$$

………………………………………………..

$$\frac{\partial^2}{\partial x^2}\Psi_n - (A_{11}\,\Psi_1 + A_{12}\,\Psi_2 \ldots\ldots\ldots\ldots A_{1n}\,\Psi_n) = \lambda\,\Psi_n$$

……………………………….. (1.2)

Let $L_x \equiv [\delta_{ij}\frac{\partial^2}{\partial x^2} - A_{ij}]$, and $L_y \equiv [\delta_{ij}\frac{\partial^2}{\partial y^2} - A_{ij}]$, where δ_{ij} is the Kronecker Delta, be two matrix operator. Then the equation (1.1) or the system of equations (1.2) can be written as

$L_x\,\Psi(x,y) - \lambda\,\Psi(x,y) = 0$ …………………………………………………[1.3(a)].

Similarly, we consider

$L_y\,\Psi(x,y) - \lambda\,\Psi(x,y) = 0$ …………………………………………………[1.3(b)].

The system (1.1) or (1.2), or [1.3(a)] and [1.3(b)] is a homogeneous system. The non-homogeneous system associated with [1.3(a)] and [1.3(b)] is

$L_x\,\Psi(x,y) - \lambda\,\Psi(x,y) = -\,f(x,y)$ and

$L_y\,\Psi(x,y) - \lambda\,\Psi(x,y) = -\,f(x,y)$,

……………………………………………………… (1.4)

Where every component of the vector $f(x,y) = [f_1(x,y), f_2(x,y), \ldots\ldots\ldots f_n(x,y)]^T$ is real valued function of x and y in $a \le x \le b$, and $c \le y \le d$.

2. Definition of the Boundary Conditions:

We define the boundary conditions to be satisfied by a vector

$\Psi(x,y) = [\Psi_1(x,y), \Psi_2(x,y), \ldots\ldots\ldots \Psi_n(x,y)]^T$ at x=a and x=b respectively by

$$a_{j1}\,\Psi_1 + a_{j2}\frac{\partial}{\partial x}\Psi_1 + a_{j3}\,\Psi_2 + a_{j4}\frac{\partial}{\partial x}\Psi_2 + \ldots\ldots + a_{j2n-1}\,\Psi_n + a_{j2n}\frac{\partial}{\partial x}\Psi_n = 0,$$

j = 1,2,……..,n ………….…………..(2.1)

and

$$b_{j1}\,\Psi_1 + b_{j2}\frac{\partial}{\partial x}\Psi_1 + b_{j3}\,\Psi_2 + b_{j4}\frac{\partial}{\partial x}\Psi_2 + \ldots\ldots + b_{j2n-1}\,\Psi_n + b_{j2n}\frac{\partial}{\partial x}\Psi_n = 0,$$

j = 1,2,……..,n ……………………..(2.2)

where

(a) a_{jk} and $b_{jk,}$ are (j=1,2,….. ,n; k = 1,2,….. ,2n) are real valued constants.
(b) The trivial case of a_{jk}= 0 for j=1,2,….. ,n, k=1,2,….. ,2n and
b_{jk}= 0 for j=1,2,….. ,n, k=1,2,….. ,2n is excluded.
(c) The sets $a_{11}, a_{12},$ ……………., a_{1n} (i=1,2,….. ,n) are linearly independent; and the sets $b_{11}, b_{12},$ ……………., b_{1n} (i=1,2,….. ,n) are linearly independent.
(d) $a_{j1}\, a_{k2} - a_{j2}\, a_{k1} + a_{j3}\, a_{k4} - a_{j4}\, a_{k3} +$ …………. $+ a_{j2n-1}\, a_{k2n} - a_{j2n}\, a_{k2n-1} = 0,\ 1\le j,k \le n$
……………………………. (2.3)

And
(e) $b_{j1}\, b_{k2} - b_{j2}\, b_{k1} + b_{j3}\, b_{k4} - b_{j4}\, b_{k3} +$ …………. $+ b_{j2n-1}\, b_{k2n} - b_{j2n}\, b_{k2n-1} = 0,\ 1\le j,k \le n$
……………………………. (2.4)

Similarly we define boundary conditions to be satisfied by a vector
$\Psi(x,y) = [\Psi_1(x,y), \Psi_2(x,y),$ ……… $\Psi_n(x,y)]^T$ at y=c and y=d respectively by

$$c_{j1}\, \Psi_1 + c_{j2}\frac{\partial}{\partial x}\Psi_1 + c_{j3}\, \Psi_2 + c_{j4}\frac{\partial}{\partial x}\Psi_2 + \ldots\ldots + c_{j2n-1}\, \Psi_n + c_{j2n}\frac{\partial}{\partial x}\Psi_n = 0,$$

j = 1,2,……..,n ………………..……….(2.5)

and

$$d_{j1}\, \Psi_1 + d_{j2}\frac{\partial}{\partial x}\Psi_1 + d_{j3}\, \Psi_2 + d_{j4}\frac{\partial}{\partial x}\Psi_2 + \ldots\ldots + d_{j2n-1}\, \Psi_n + d_{j2n}\frac{\partial}{\partial x}\Psi_n = 0,$$

j = 1,2,……..,n ……………………….(2.6)

where
(a) c_{jk} and $d_{jk,}$ are (j=1,2,….. ,n; k = 1,2,….. ,2n) are real valued constants.
(b) The trivial case of c_{jk}= 0 for j=1,2,….. ,n, k=1,2,….. ,2n and
 a. d_{jk}= 0 for j=1,2,….. ,n, k=1,2,….. ,2n is excluded.
(c) The sets $c_{11}, c_{12},$ ……………., c_{1n} (i=1,2,….. ,n) are linearly independent; and the sets $d_{11}, d_{12},$ ……………., d_{1n} (i=1,2,….. ,n) are linearly independent.
(d) $c_{j1}\, c_{k2} - c_{j2}\, c_{k1} + c_{j3}\, c_{k4} - c_{j4}\, c_{k3} +$ …………. $+ c_{j2n-1}\, c_{k2n} - c_{j2n}\, c_{k2n-1} = 0,\ 1\le j,k \le n$

……………………….. (2.7)
(e) $d_{j1}\, d_{k2} - d_{j2}\, d_{k1} + d_{j3}\, d_{k4} - d_{j4}\, d_{k3} +$ …………. $+ d_{j2n-1}\, d_{k2n} - d_{j2n}\, d_{k2n-1} = 0,\ 1\le j,k \le n$

……………………….. (2.8)

The conditions (2.3), (2.4), (2.7), and 2.8 are necessary for making the system orthogonal.

3. Eigenvalue and Eigenvector:

As usual the vector $\Psi \equiv \Psi(x,y) = [\Psi_1(x,y), \Psi_2(x,y),$ ……… $\Psi_n(x,y)]^T$ which satisfies [1.3(a)] and [1.3(b)] is a solution of the system [1.3(a)] and [1.3(b)] and the solution $\Psi \equiv \Psi(x,y) = [\Psi_1(x,y), \Psi_2(x,y),$ ……… $\Psi_n(x,y)]^T$ such that $\Psi_1(x,y)= \Psi_2(x,y)=$ ……… $=\Psi_n(x,y = 0$ for all x,y in a≤x≤b, and c≤y≤d, is a trivial solution. We exclude such a solution from our discussion.
The value of λ for which the system [1.3(a)] and [1.3(b)] has a non-trivial solution satisfying other conditions is an eigenvalue, and the corresponding vector is an eigenvector.

4. Existence Theorem:

Let $A_1, A_2,$ ………. , A_{2n} be 2n constants. Then the equations [1.3(a)] and [1.3(b)] has a solution $\phi(x,y) = [\phi_1(x,y), \phi_2(x,y),$ ……… $\phi_n(x,y)]^T$ which is unique and which satisfies $\phi_1(x,y)= A_1, \frac{\partial}{\partial x}\phi_1(x,y)= A_2, \phi_2(x,y)= A_3, \frac{\partial}{\partial x}\phi_2(x,y)= A_4,$…………….. $\phi_n(x,y)= A_{2n-1}, \frac{\partial}{\partial x}\phi_n(x,y)= A_{2n},$ ……………………………………………………. (4.1)

The proof follows by using the method of Bhagat (1).

5. Bilinear Concomitant:

Let $\phi_j= \phi_j(x,y)= [\phi_{j1}(x,y), \phi_{j2}(x,y),$ ……… $\phi_{jn}(x,y)]^T$ and
$\phi_k= \phi_k(x,y)= [\phi_{k1}(x,y), \phi_{k2}(x,y),$ ……… $\phi_{kn}(x,y)]^T$ where ϕ_{j1}, ϕ_{k1} etc.are functions of (x,y) be two eigenvectors. Then the bilinear concomitant of the two vectors ϕ_j and ϕ_k is defined by
$\frac{\partial}{\partial x}\phi_{j1}\ \phi_{k1}+ \frac{\partial}{\partial x}\phi_{j2}\ \phi_{k2}+$………… $\frac{\partial}{\partial x}\phi_{jn}\ \phi_{kn}+\ \phi_{j1}\frac{\partial}{\partial x}\phi_{k1}-\ \phi_{j2}\frac{\partial}{\partial x}\phi_{k2}$ -……….. $-\ \phi_{jn}\frac{\partial}{\partial x}\phi_{kn}$ and represented by $[\phi_j\ \phi_k]_x$

and
$\frac{\partial}{\partial y}\phi_{j1}\ \phi_{k1}+ \frac{\partial}{\partial y}\phi_{j2}\ \phi_{k2}+$………… $\frac{\partial}{\partial y}\phi_{jn}\ \phi_{kn}+\ \phi_{j1}\frac{\partial}{\partial y}\phi_{k1}-\ \phi_{j2}\frac{\partial}{\partial y}\phi_{k2}$ -……….. $-\ \phi_{jn}\frac{\partial}{\partial y}\phi_{kn}$ and represented by $[\phi_j\ \phi_k]_y$. The following properties of the bilinear concomitant are evident:
(i) $[\phi_j\ \phi_j]_x = 0$

(ii) $[\phi_j \ \phi_j]_x = - [\phi_k \ \phi_j]_x$

(iii) $[\phi_j \ (r\phi_u + s\phi_v]_x = r[\phi_j \ \phi_u]_x + s[\phi_j \ \phi_v]_x$

Similar results hold for $[\phi_j \ \phi_j]_x$.

6. Conclusion:

A matrix partial differential operator has been defined with suitable boundary conditions. Bilinear concomitant has been obtained and some important properties have been proved.

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