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On Product of Hypergeometric Function, General Class of Multivariable Polynomials and a Generalized Hypergeometric Series Associated with Feynman Integrals

Rakesh Ranjan

(Research Scholar) Department of Mathematics, V.K.S. University, Ara , Bihar

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ABSTRACT

In this paper, we first establish a finite integral and then obtain three theorems with the help of this integral and the results given by Slater (1966). These results involve the product of hypergeometric functions, two classes of multivariable polynomials and the $\bar{H}$-function introduce by Inayat-Hussain (1987) and further studied by Buschman an Srivastava (1990). This $\bar{H}$-function contains certain class Feynman integrals, the exact partition function of the Gaussian model in statistical mechanics, the polylogarithm of order p and several other functions as its particular cases. The results obtained here are quite general in nature due to the presence of two general classes of multivariable polynomials and the $\bar{H}$-function, and we can derive from them a number of similar results involving simpler functions and polynomials. For the same of illustration, we have mentioned here three particular cases of our results which are new and of interest by themselves. References of some known results which follow from the main finding of this paper have also been given.

1. Introduction

Feynman path integrals are reformulation of quantum mechanics and are more fundamental then the conventional one in terms of operators because in the domain of quantum cosmology, the conventional formulation may fail but Feynman path integrals still apply. Inayat-Hussain (1987a) while demonstrating the usefulness of Feynman integrals in enabling the derivation of new transformation, summation and reduction formulae for multiple hypergeometric series was led to a generalized function which he denoted by $\bar{H}$ (Inayat-hussain 1987b). This $\bar{H}$-function is a generalization of the familiar Fox's H-function and is defined as follows

(1)
$$\bar{H}^{M,,N}_{P,Q}[z] = \bar{H}^{M,,N}_{P,Q}\left[z \middle| \begin{matrix} (a_j,\alpha_j,A_j)_{l,N,}(a_j,\alpha_j)_{N+l,P} \\ (b_j,\beta_j)_{l,M,,}(b_j,\beta_j;B_j)_{M+l,Q} \end{matrix} \right]$$

$$= \frac{1}{2\pi\omega}\int_{-\omega\,\infty}^{\omega\,\infty} \bar{\phi}(\xi) z^{\xi} d\xi \quad \omega = \sqrt{-1}$$

(2)
$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{M}\Gamma(b_j\beta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-a_j+\alpha_j\xi)\}\Lambda_j}{\prod_{j=M+1}^{Q}\{(1-b_j+\beta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(a_j-\alpha_j\xi)}$$

Here a_j (j = 1, ….., P) and b_j (j = 1,……,Q) are complex parameters, $a_j \geq 0$ (j = 1,……,P) and $\beta_j \geq 0$ (j = 1,…..,q) (not all zero simultaneously) and the exponents A_j (j = 1,……,N) and B_j (j = M + 1, …..,Q) can take on non-integer values. For the absolute convergence of the $\bar{H}$-function, the following sufficient conditions are given by Buschman and Srivastava (1990)

(3)
$$\Omega = \sum_{j=1}^{M}\beta_j = \sum_{j=M+1}^{Q}\beta_j B_j + \sum_{j=1}^{N}A_j\alpha_j - \sum_{j=N+1}^{P}\alpha_j > 0$$

(4)
$$|\arg(z)| < \frac{1}{2}\pi\Omega$$

and

When all the exponents A_j and B_j take on integer values, the $\bar{H}$-function reduces to the well known Fox's H-function (Fox 1961).

We given below few particular cases of the $\bar{H}$-funciton which are not the spedl cases of foxs H-function.

(i) $g_1 = (-1)^p g(e, \eta, f, p; z)$

$$= \frac{K_{d-1}\Gamma(P+1)\Gamma\left(1+\frac{1}{2}\right)B\left(\frac{1}{2},\frac{1}{2}+\frac{1}{2}\right)}{2^{p+2}\pi\Gamma(e)\Gamma\left(c-\frac{1}{2}\right)}\frac{1}{2}\pi\omega\int_{-\omega\infty}^{\omega\infty}\frac{(-z)^{\xi}\infty(-\xi)\Gamma(e+\xi)\Gamma\left(e-\frac{f}{2}+\xi\right)}{(\eta+\xi)^{1+p}\Gamma\left(1+\frac{f}{2}+\xi\right)}$$

(5)

$$= \frac{\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}{2^{p+d}\pi^{d/2}\Gamma\left(\frac{d-1}{2}\right)\Gamma(e)\Gamma\left(e-\frac{f}{2}\right)}$$

$$\bar{H}_{3,3}^{1,3}\left[-z\left[\begin{matrix}(1-e,1;1)\left(1-e+\frac{f}{2},1;1\right),(1-\eta,l;p+1)\\(0,1),\left(-\frac{f}{2},1;1\right),(-\eta,1lp+1)\end{matrix}\right]\right]$$

where

$$K_{d-1} = \frac{2^{2-d}\pi^{(1-d)/2}}{\Gamma(d-1)/2}$$

The above function is connected with a certain class of Feynman integrals (Inayat-Hussain 1987 b, p, 4121, eq.1)

Further if we taken $e = 1+\frac{f}{2}$ in (1), we get

$$g\left(1+\frac{f}{2},\eta,f,p;z\right)$$

$$= \frac{2^{-p-d}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{f}{2}\right)}{(-1)^p\pi^{d/2}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(1+\frac{f}{2}\right)}\bar{H}_{2,2}^{1,2}\left[-z\left[\begin{matrix}(0,p,1),(1-\eta,k,p+1)\\(0,1),(-\eta,l;p+1)\end{matrix}\right]\right]$$

(6)

Which can be written as

$$g\left(1+\frac{f}{2},\eta,f,p;z\right) = \frac{2^{-p-d}\Gamma(p+1)\Gamma\left(\frac{1}{2}+\frac{f}{2}\right)}{(-1)^p\pi^{d/2}\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(1+\frac{f}{2}\right)}\phi(z,\rho+l,\eta)$$

(7)

where $\phi(z,p,\eta)$ is a generalized Riemann zeta function (Erdelyi 1953, p. 27, 1.11 (1)) which is a generalization of the well known generalized Hurwitz's zeta function $\zeta(p,\eta)$ and Riemann zeta function $\zeta(p)$ (Erdelyi 1953, p. 24, eq. 10(1), p. 32, eq. 12(1)).

(ii) $\beta' F(d, \varepsilon($

(8)
$$= \frac{-(1+\varepsilon)^{-2}}{4\pi^{d/2}} \frac{1}{2\pi\omega} \int_{-\omega\infty}^{\omega\infty} \frac{\left[-(1+\varepsilon)^{-2}\right]^{\xi} \Gamma(-\xi)\{\Gamma(1+\xi)\}^{2}\{\Gamma(3/2+\xi)\}^{d}}{\{\Gamma(2+\xi)\}^{l+d}} d\xi$$

$$= \frac{-(1+\varepsilon)^{-2}}{4\pi^{d/2}} \bar{H}_{3,3}^{1,3}\left[-(1+\varepsilon)^{-2}\left| \begin{matrix} (0,1;1),(0,1;1),1-e,1;1),\left(-\frac{1}{2},l;d\right) \\ (0,1),(-l,l;l+d) \end{matrix} \right.\right]$$

The above function is the exact partition function of the the Gaussian model in statistical mechanics (Inayat-Hussain 1987b, p. 4217, eq. 28).

(iii) The Polylogarithm of order p (Erdelyi 1953, p.30, 1.11) (14))

(9)
$$F(z,p) = \sum_{n+1}^{\infty} \frac{z^{A}}{\eta^{p}} = \bar{H}_{2,3}^{1,2}\left[-z\left| \begin{matrix} (1,1;1),\ (l,l;p) \\ (1,1),(0,l;p) \end{matrix} \right.\right]$$

For p=2, the above function reduces into Euler's dilogarithm (Erdeli 1953, p. 31, 1.11.1(22)).

We shall also require the following two classes of multivariable polynomials. The first class of multivariable polynomials introduced by Srivastava and Garg [1987) is defined in the following form :

(10)
$$S_{F}^{r},p(z_{i}.....z_{r}] = \sum_{l_1,k_r+0}^{p_1\lambda_1+\ \ +l/\lambda,d} (-V)U_{l}k_{l} + ... + U,k,$$

$$A(V;k_{1}.......k_{r})\frac{z_{1}^{l_1}}{k_{1}^{l}}$$

Where U_{l},……U_{r} are arbitrary positive integers and the coefficient

$$a(V;k_{1}......k_{r})\ \ (V,k,\geq 0,\ \ i = l,........,r)$$

being arbitrary constant, real or complex.

The second class of multivariable polynomials given by Srivastava (1972) is defined and represented in the slightly modified form as follows :

(11)
$$\sum_{(k_1-1)}^{P_1F_1} .... \sum_{(k_1-1)}^{P_1F_1} ....(-F)U_{1}k_{1}.....(V_{1}),A^{'}(P_{1},k_{1}.....;P,k_{r})\frac{z_{1}^{l_1}}{k_{1}^{l}}.....\frac{z_{1}^{l_1}}{k_{r}}$$

Where U_1….$U_,$ are arbitrary positive integers, the coefficient $(\Gamma_{1}l_{1};....:\Gamma,\lambda)(\Gamma_{r},k_{r})(\Gamma_{r},k_{r} \geq 0, l = 1....r)$ being arbitrary constant, real or complex.

2. A useful Integral : In this section we shall obtain the following finite integral involving the product of the first and second class of multivariable polynomials and the $\bar{H}-$function defined by (10), (11) and (2.5.1) respectively.

$$\int_{0}^{b} z^{\rho-1}(b-z)^{a-1} S_{r_{1}.......V_{r}}^{U_{v}.......U_{r}}\left[X_{1}z^{\mu_1}(b-z)^{v_1}......X_{r},z^{\mu_d}(b-z)v_{1}\right]$$

$$\times S_{V}^{U},.....U_{1}\left[X_{r+1}z^{u_r+1}(b-z)^{v_r+1}......,X_{s},z^{\mu}(b-z)v_{1}\right]$$

$$\times\bar{H}_{P,Q}^{M,N}\left[Yz^{\mu}(b-z)^{v}\left| \begin{matrix} (\alpha_{j},\alpha_{j};A_{j})_{1,x},(\alpha_{j},\alpha_{j})x+1.P \\ (b_{j},\beta_{1})_{1,M,}(b_{1},\beta_{j};B_{j})_{M+1,Q} \end{matrix} \right.\right]dz$$

$$= b^{p+\sigma-1} \sum_{k_1.....l_r} A^i(V_1,k_1.....,V_r,k_r)A(V;k_{r+1}....,k_x)\prod_{i=1}^{r}(-V_i)U_ik_i$$

$$\times(V)_{U_{r+1}l_{r+1}+....+U_2^1l_1} \prod_{l=1}^{3}\left\{\frac{x_l^{l_1}}{k_l^1} b^{(\mu)+v_1)H_1}\right\}\bar{H}^{M,N+2}_{l+2(1+1)}$$

$$\left[Yb^{\mu+v}\left|\left(1-\rho-\sum_{l=1}^{s}\mu_1 k_1,\mu;1\right)\left(1-\sigma-\sum_{l=1}^{s}V_1 k_1,v;1\right),(a_j,\alpha_j;A_j)_{1,N}(a_j,\alpha_j)_{N+1,l}\right.\right]$$

$$(b_j,\beta_j)_{1,M,}(b_j,\beta_j;B_j)_{M+1,Q}\left[1-\rho-\sigma-\sum_{l=1}^{s}(\mu_1+v_1)k_1,\mu+v;1\right]$$

Where $\sum_{k_1-l_p}$ stands for $\sum_{l_1=0}^{[V_1/U_1]}......\sum_{k_r=0}^{[V_r/U_r]}\sum_{k_{r+1}.......,k_1=0}^{U_{r+1}k_{r+1}+.....+U_3l_s\le l}$

The result hold true under the following conditions :

(i) $\mathrm{Re}(\rho,\sigma)>0$

(ii) $\min(\mu,v,\mu_1,v_1)\ge 0,(l=1,....,s)$

(iii) $\mathrm{Re}(\rho)+\mu\min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)]>0$

$\mathrm{Re}(\sigma)+v\min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)]>0$

(iv) The $\bar{H}$ - function occurring in (1.1) satisfies the conditions corresponding to those given by (3) and (4).

Proof : the above integral can easily be established on using the Bellin-Barnes contour integral for $\bar{H}$ -function, series form of multivariable polynomials and the well known Eulerian Beta integral.

On taking s = r + 1 and r = 1 in the above integral, we easily arrive at a result obtained recently by Gupta and Soni (2001)

Main Results

THEOREM 1: If $(1-z)^{\alpha+\beta-\gamma-1/2}\,{}_2F_1[2\alpha,2\beta;2\gamma;z]=\sum_{n=0}^{\infty}a_n z^n$, then

$$\int_0^b z^{\rho-1}(b-z)^{\sigma-1}\,{}_2F_1[\alpha,\beta;\gamma;z]\,{}_2F_1[\gamma-\alpha+1/2,\gamma-\beta+1/2;\gamma+1;z]$$

$$\times S^{U_i----U_r}_{r_1------r_r}[X_2 z^{\mu_1}(b-z)^{r_1}......,X_r,z^{\mu_r}(b-z)^{v_r}]$$

$$\times S^{U_{r+1}----U_r}_{r}[X_{r+1}z^{\mu_r+1}(b-z)^{r_r+1},....,X_s,z^{\mu_s}(b-z)^{v_r}]$$

$$\times\bar{H}^{M,N}_{P,Q}\left[z^{\mu}(b-z)^{r}\left|\begin{matrix}(\alpha_j,\alpha_j;A_j)_{1,N},(\alpha_j,\alpha_j)_{N-1.P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_{j;}B_j)_{M-1,Q}\end{matrix}\right.\right]dz$$

$$= b^{\rho+\sigma-1} \sum_{k_1....k_r} \sum_{N=0}^{\infty} A'(V_1,k_1;.....,V_{r,}k_r)A(V;k_{r+1,}....,k_3)\prod_{i=1}^{r}(-V_i)U_i k_i$$

$$\times(-v)_{L_{r+1}l_{r+1}+.....U_r l_r} \prod_{l=1}^{s}\left\{\frac{x_l^{k_i}}{k_l^i} b^{p_l-r_l)l_l}\right\}\frac{(\gamma+1/2)_n}{(\gamma+1)_r} a_n b^n \bar{H}_{P-2Q-1}^{M,N-2}$$

$$\left[Yb^{\mu+1} \left| \begin{matrix} \left(1-\rho-n-\sum_{l=1}^{s}\mu_l k_l,\mu;1\right),\left(1-\sigma-\sum_{l=1}^{s}v_l k_l,v;l\right)(a_j,\alpha_j;A_j)_{l,N}(a_j,\alpha_j)_{N+1} \\ (b_j,\beta_j)_{l,M,}(b_j,\beta_j;B_j)_{M+1,Q}\left(1-\rho-\sigma-n-\sum_{l=1}^{s}(\mu_1+v_1)k_1,\mu+v;l\right) \end{matrix} \right. \right.$$

where $\sum_{k_1.......k_n}$ stands for $\sum_{k_i=0}^{(V_1/U_1]} ..... \sum_{k_r=0}^{[V_r/U_r]} \sum_{k_{r+1}.......k_r=0}^{U_r,k_{r+1}+.....+U_r k_i \le 1}$ and the following conditions are assumed to be satisfied.

(i) $\mathrm{Re}(\rho,\sigma) > 0$

(ii) $\min(\mu, v, \mu_1, v_1) \ge 0, (l = 1,....,s)$ (not all zero simultaneously)

(iii) $\mathrm{Re}(\rho) + \mu \min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)] > 0$

$\mathrm{Re}(\sigma) + v \min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)] > 0$

The $\bar{H}-$function occurring in (2.1) satisfies the conditions corresponding to those given by (3) and (4)

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