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# A Cubic Spline Formulation and Application of Newton Tage Method for the Solution of Two-Point Boundary Value Problems

[1]Seema and [2]Dr. Ankush

[1]Research Scholar, Department of Mathematics, NIILM University, Kaithal
[2]Asstt. Professor, Department of Mathematics, NIILM University, Kaithal

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**ABSTRACT**

*In this article, we report a proficient high order numerical method dependent on cubic spline estimate and use of newton TAGE strategy for the arrangement of two point non-linear boundary esteem issues, whose compelling capacities are in vital structure, on a non-uniform mesh. The proposed strategy is relevant when the inward network purposes of arrangement span are odd in number. The proposed cubic spline strategy is likewise material to necessary differential conditions having singularities. Computational outcomes are given to show the utility of the strategy.*

**Introduction: -**

Disregarding the way that splines can be of any degree, cubic splines are by a wide edge the most standard. A cubic spline is a spline created of piecewise third-mastermind polynomials which experience a great deal of control centers. The second subordinate of each polynomial is normally set to zero at the endpoints, since this gives a limit condition that completes the arrangement of conditions. This makes a claimed "natural‟ cubic spline and prompts a clear tri-corner to corner framework which can be handled easily to give the coefficients of the polynomials. Regardless, this choice isn't the only a solitary possible and other limit conditions may be used.

**Cube Spline Formulation**

We discretize arrangement locale [0,1] with non-uniform work so such an extent that $0 = x_0 < x_1 \dots\dots < x_{N+1} = 1$ . Our method includes three matrix focuses $x_k, x_{k+1} \& x_{k-1}$where$x_k - x_{k-1} = h_k \& x_{k+1} - x_k = h_{k+1}$. Matrix focuses are given by $x_i = x_0 + \sum_{k=1}^{i} h_k$ , $i = 1(1)N + 1$and the work extent is $\sigma_K = (h_{k+1}/h_k) > 0$. Right when $\sigma_K = 1$ , at that point it decreases to constant mesh case. Let the right estimations of $u(x)$ at matrix point $x_k$be implied by $U_K = u(x_k)$and $u_k$be the rough estimations of $U_K$.

At the matrix point $x_k$, let us show

$P_k = \sigma_k^2 + \sigma_k - 1$

$Q_k = (\sigma_k + 1)(\sigma_k^2 + 3\sigma_k - 1)$

$R_k = \sigma_k(1 + \sigma_k - \sigma_k^2$

$S_k = \sigma_k(\sigma_k + 1)$

All through our trade, we consider that our answer contains odd number of inside grid points, for instance$N$ as odd.

We initially build up a numerical method for evaluating integral$\int_0^1 G(x)dx$. For this we consider

$$\int_{x_{k-1}}^{x_{k+1}} G(x)\,dx = b_{-1}G_{k-1} + b_0 G_k + b_1 G_{k+1} \qquad (1)$$

Where $b_{-1}, b_0$ , $b_1$ are boundaries to be chose and at point $x_k$ , we show$G_k = G(x_k)$

Further, we may compose utilizing Taylor's expansion

$$G_{k-1} = G_k - h_k G_k' + \frac{h_k^{\,2}}{2} G_k'' - \frac{h_k^{\,3}}{6} G_k''' + \dots\dots\dots \qquad (2)$$

$$G_{k+1} = G_k + h_{k+1} G_k' + \frac{h_{k+1}^{\,2}}{2} G_k'' + \frac{h_{k+1}^{\,3}}{6} G_k''' + \dots\dots\dots \qquad (3)$$

$$\int_{x_{k-1}}^{x_{k+1}} G(x)\,dx = \int_{x_{k-1}}^{x_{k+1}} G(x_k + x - x_k)\,dx$$

$$= (1 + \sigma_k) h_k G_k + (\sigma_k^2 - 1)\frac{h_k^2}{2} G_k' + (1 + \sigma_k^3)\frac{h_k^3}{6} G_k''$$

$$+ (\sigma_k^4 - 1)\frac{h_k^4}{24} G_k''' + \cdots \qquad (4)$$

Likewise right hand side of (1), utilizing (2), (3) as

$$b_{-1}\left[G_K - h_K G_K^{'} + \frac{h_k^2}{2} G_k^{"} + \frac{h_k^4}{24} G_K^{IV} - ..\right] + b_0 G_k$$

$$+b_1\left[G_K + \sigma_k h_K G_K^{'} + \frac{\sigma_k^2 h_k^2}{2} G_k^{"} + \frac{\sigma_k^3 h_k^3}{6} G_k^{"} + \frac{\sigma_k^4 h_k^4}{24} G_K^{IV} + ..\right] \quad (5)$$

Using (4)-(5), comparing both sides of (1), we get

$b_0 + b_1 + b_{-1} = (1+\sigma_k)h_k$ (6)

$(b_1\sigma_k - b_{-1})h_k = (\sigma_k^2 - 1)\frac{h_k^2}{2}$ it implies $b_1\sigma_k - b_{-1} = (\sigma_k^2 - 1)\frac{h_k}{2}$.(7)

$b_1\sigma_{k\,b_{-1}}^2 = (\sigma_k^3 - 1)\frac{h_k}{3}$ (8)

Solving (6)-(8), we get

$$b_0 = \frac{(1+\sigma_k)^3}{6\sigma^k} h^k, b_1 = \frac{(1+\sigma_k)(2\sigma_k - 1)}{6\sigma^k} h^k, b_{-1} = \frac{(1+\sigma_k)(2-\sigma_k)}{6} h_k$$

Accordingly one might be compose third request integral formula (2) in variable mesh frame as

$$\int_{x_{k-1}}^{x_{k+1}} G(x)dx = \frac{(1+\sigma_k)h_k}{6\sigma_k}\left[\sigma_k(2-\sigma_k)G_{k-1} + (1+\sigma_k)^2 G_k + (2\sigma_k - 1)G_{k+1}\right],$$

$$k = 1(1)N \quad (9)$$

Note that coefficients related with right hand side of (9) are sure, assuming likewise, for example, $2-\sigma_k > 0\&2\sigma_k - 1 > 0$ i.e $\frac{1}{2} < \sigma_k < 2$ . Regardless, for constant work case, integral formulas (9) diminish to Simpson's 1/third Rule.

By then the estimation of integral

$$\int_0^1 G(x)dx = \int_{x_0}^{x_2} G(x)dx + \int_{x_2}^{x_4} G(x)dx + ...... + \int_{x_{N-1}}^{x_{N+1}} G(x)dx \quad (10)$$

Can be found by the reiterated use of formula (2), when N is odd

Directly we analyzed the variable work cubic spline strategy for differential con. (3). Cubic spline interjecting polynomial of degree three then$[x_{k-1,}x_k]$ can be made as

$$S(x) = \frac{(x_k - x)^3}{6h_k} M_{k-1} + \frac{(x - x_{k-1})^3}{6h_k} M_k + (u_{k-1} - \frac{h_k^2}{6} M_{k-1})\frac{(x_k - x)}{h_k}$$

$$+(u_k - \frac{h_k^2}{6} M_k)(\frac{x - x_{k-1}}{h_k}) \ , \quad x_{k-1} \le x \le x_k \ , \ k = 1(1)N+1 \quad (11)$$

Which satisfies the accompanying with conditions

(i) in each subinterval $[x_{k-1,}x_k], k = 1(1)N+1, S(x)$, coincides with a polynomial of degree three,

(ii) $S(x) \in C^2[0,1]$ and

(iii) $S(x_k) = u_k, \quad k = 0(1)N+1$

Where $M_k = u^{"}(x_k) = \psi(x_k, u(x_k), u^{'}(x_k)) \equiv say$

And $m_k = u^{'}(x_k), k = 0N+1$

Now,

$$u(x) \cong S(x) = \frac{(x_{k+1}-x)^3}{6h_{k+1}} M_k + \frac{(x-x_k)^3}{6h_{k+1}} M_{k+1} + \left(u_k - \frac{h_{k+1}^2}{6} M_k\right)\left(\frac{x_{k+1}-x}{h_{k+1}}\right)$$

$$+\left(u_{k+1} - \frac{h_{k+1}^2}{6} M_{k+1}\right)\left(\frac{x-x_k}{h_{k+1}}\right) \ , \quad x \in [x_k, x_{k+1}]$$

$$S^{'}(x) = -\frac{(x_{k+1}-x)^2}{2h_{k+1}} M_k + \frac{(x-x_k)^2}{2h_{k+1}} M_{k+1} + \frac{u_{k+1}-u_k}{h_{k+1}} - \frac{h_{k+1}}{6}[M_{k+1} - M_k], \quad x \in [x_k, x_{k+1}]$$

From continuity con.$S^{'}(x_{k-}) = S^{'}(x_{k+})$

i.e.$\lim_{\in\to 0} S^{'}(x_k - \in) = \lim_{\in\to 0} S^{'}(x_k + \in)$ one obtains

$$\frac{h_k}{6} M_{k-1} + \frac{h_k + h_{k+1}}{3} M_k + \frac{h_{k+1}}{6} M_{k+1}$$

$$= \frac{u_{k+1}-u_k}{h_{k+1}} - \frac{u_k - u_{k-1}}{h_k} \ , \ k = 1(1)N$$

Where $M_k = u^{"}x_k, M_{k+1} = u^{"}x_{k+1}M_{k-1} = u^{"}x_{k-1},$

Here one has equations in questions $N+2$ unknowns $M_0, M_1, M_2, ... M_N, M_{N+1}$

If $M_0 = u^{"}x_0$ and $M_{N+1} = u^{"}x_{N+1}$ are use then one can calculate $M_1, M_2 \ .... M_N,$

$\bar{M}_k = \psi(x_k, U_k, \bar{m}_k,$ (12)

$\bar{M}_{k+1} = \psi(x_{k+1}, U_{k+1}, \bar{m}_{k+1}$ (13)

$\bar{M}_{k-1} = \psi(x_{k-1}, U_{k-1}, \bar{m}_{k-1}$ (14)

$S(x_{k+1}) = U_{K+1}$

And

$$S'(x_{k+1}) \cong m_{k+1} = \frac{U_{K+1} - U_k}{h_{k+1}} + \frac{h_{k+1}}{6}[M_K + 2M_{K+1}]$$

$$S'(x_{k-1}) \cong m_{k-1} = \frac{U_K - U_{k-1}}{h_k} + \frac{h_k}{6}[M_{K-1} + M_K]$$

Define $G = \frac{\partial\psi}{\partial u}$, $G' = \frac{dG}{dx}$ etc.

We require accompanying approximations

$M_k \simeq \bar{M}_k = \psi(x_k, U_k, \bar{m}_k)$

$= \psi_k + \frac{1}{6}\sigma_k h_k^2 U_K^" G_k + O(h_k^3)$

$M_{k+1} \simeq \bar{M}_{k+1} = \psi(x_{k+1}, U_{k+1}, \bar{m}_{k+1})$

$= \psi_{k+1} - \frac{1}{6}\sigma_k(\sigma_k + 1)h_k^2 U_K^" G_k + O(h_k^3)$

$M_{k-1} \simeq \bar{M}_{k-1} = \psi(x_{k-1}, U_{k-1}, \bar{m}_{k-1})$

$= \psi_{k-1} - \frac{1}{6}\sigma_k(\sigma_k + 1)h_k^2 U_K^" G_k + O(h_k^3)$

One need $O(h_k^3)$ approximation for $\hat{U}_k^l$ and $\hat{U}_{k\pm1}^l$.

Let, $\hat{U}_k^l = \bar{m}_k + a_k h_k[\bar{M}_{K+1} - \bar{M}_{K-1}]$ where "$a_k$"is a parameter to be considered.

$= m_k + (\frac{1}{6}\sigma_k + a_k(\sigma_k + 1)h_k^2 U_K^" + O(h_k^3)$ (15)

The approximation (15) to be of, coefficient of must be zero;

$$a_k = -\frac{\sigma_k}{6(1 + \sigma_k)}$$

Hence one obtain

$$\hat{U}_k^l = \bar{m}_k - \frac{\sigma_k}{6(1+\sigma_k)} h_k[\bar{M}_{k+1} - \bar{M}_{k-1}] = m_k + O(h_k^3), \quad (16)$$

Similarly,

$$\hat{U}_{k+1}^l = \frac{U_{k+1} - U_k}{\sigma_k h_k} + \frac{\sigma_k h_k}{6}[\bar{M}_k + 2\bar{M}_{k+1}] = m_{k+1} + O\left(h_k^3\right) \quad (17)$$

$$\hat{U}_{k-1}^l = \frac{U_k - U_{k-1}}{h_k} - \frac{h_k}{6}[2\bar{M}_{k-1} + \bar{M}_k] = m_{k-1} + O\left(h_k^3\right) \quad (18)$$

Finally, by assistance of (16)- (18), one can assess

$\hat{\psi}_{k+1} = \psi\left(x_{k+1}, U_{k+1}, \hat{U}_{k+1}^l\right) = \psi_{k+1} + O(h_k^3)$

$\hat{\psi}_{k-1} = \psi\left(x_{k-1}, U_{k-1}, \hat{U}_{k-1}^l\right) = \psi_{k-1} + O(h_k^3)$

$\hat{\psi}_k = \psi\left(x_k, U_k, \hat{U}_k^l\right) = \psi_k + O(h_k^3)$

By then the cubic spline approximation of precision of $O(h_k^3)$for given integral differential conditions (1) may be created as

$$U_{k+1} - (1+\sigma_k)U_k + \sigma_k U_{k-1} = \frac{h_k^{\ 2}}{12}[P_k\hat{\psi}_{k+1} + Q_k\hat{\psi}_k + R_k\hat{\psi}_{k-1}] + \hat{T}_k, \quad k=1(1)N \quad (19)$$

Where $\hat{T}_k = O(h_k^5)$

For convergence of the distinction method, coefficients onright hand side of conditions (19) must be sure, i.e., one has the condition $\frac{\sqrt{5}-1}{2} < \sigma_k < \frac{\sqrt{5}+1}{2}$forced on our choice of mesh proportion parameter.

Note that the limitesteems are given by $u_0 = U_0 = a_0 \& u_{N+1} = U_{N+1} = a_1$ . Joining the limit esteems in numerical approximation (19), we can procure a tri-diagonal framework of equations. If the differential equation is linear, one can comprehend the framework by AGE iterative strategy and for nonlinear case, one can compared the framework by Newton-AGE iterative method.

**Application of Newton TAGE Method**

Next we talk about the utilization of Newton-TAGE iterative strategy for nonlinear difference conditions (19). Dismissing the mistake term, one may re-compose nonlinear difference equation (19) as

$$\phi_k(u_{k-1}, u_k, u_{k+1}) \equiv -u_{k+1} + (1+\sigma_k)u_k - \sigma_k u_{k-1} + \frac{h_k^{\ 2}}{12}[P_k\hat{\psi}_{k+1} + Q_k\hat{\psi}_k + R_k\hat{\psi}_{k-1}], \quad (20)$$

$$k = 1(1)N, \qquad \sigma_K \neq 1$$

Let us denote,

 

$$\boldsymbol{\phi}(\boldsymbol{u}) = \begin{bmatrix} \phi_1(u) \\ \phi_2(u) \\ \vdots \\ \phi_N(u) \end{bmatrix}_{N\times 1}$$

And,

$a_k(u) = \dfrac{\partial \phi_k}{\partial u_{k-1}}$ $k = 2(1)N,$

$2b_k(u) = \dfrac{\partial \phi_k}{\partial u_k}$ $k = 1(1)N$

$c_k(u) = \dfrac{\partial \phi_k}{\partial u_{k+1}}$ $k = 1(1)N-1$

The Jacobian of $\emptyset(u)$ may be spoken as

$$(\boldsymbol{T}_1 + \boldsymbol{T}_2) \equiv \frac{\partial \boldsymbol{\phi}(\boldsymbol{u})}{\partial u} = \begin{bmatrix} 2b_1(u) & c_1(u) & & & \boldsymbol{0} \\ a_2(u) & 2b_2(u) & c_2(u) & & \\ & & \ddots & & \\ & & a_{N-1}(u) & 2b_{N-1}(u) & c_{N-1}(u) \\ \boldsymbol{0} & & & a_N(u) & 2b_N(u) \end{bmatrix}_{N\times N}$$

Newton method is given by

$(T_1 + T_2)\Delta u^{(s)} = \Delta(u^{(s)}), s = 0,1,2 \ldots$ (21)

where $T_1$&$T_2$ are of indistinct casing from $A_1$,$A_2$ , &$u^{(0)}$ishidden estimation for $u$&$\Delta u^{(s)}$ is any halfway vector.

We characterize

$u^{(s+1)} = u^{(s)} + \Delta\, u^{(s)}, s = 0,1,2 \ldots\ldots$ (22)

By then a two-advance explicit Newton-TAGE strategy may be formed as

$$(\boldsymbol{T}_1 + \rho_1 \boldsymbol{I})\, \boldsymbol{\Delta v} = -\boldsymbol{\phi}(\boldsymbol{u}^{(s)}) - (\boldsymbol{T}_2 - \rho_1 \boldsymbol{I})\, \boldsymbol{\Delta u}^{(s)}, \quad s \geq 0 \quad (23)$$

$$(\boldsymbol{T}_2 + \rho_2 \boldsymbol{I})\, \boldsymbol{\Delta u}^{(s+1)} = -\boldsymbol{\phi}(\boldsymbol{u}^{(s)}) - (\boldsymbol{T}_1 - \rho_2 \boldsymbol{I})\, \boldsymbol{\Delta v}, \quad s \geq 0 \quad (24)$$

where $\Delta v$ is any moderatevector

Since the matrices $(T_1 + \rho_1 I)$&$(T_2 + \rho_2 I)$comprise of (2×2) sub-matrices, hereafter they are successfully invertible and from (23) & (24), we get

$$\boldsymbol{\Delta v} = (\boldsymbol{T}_1 + \rho_1 \boldsymbol{I})^{-1}[-\boldsymbol{\phi}(\boldsymbol{u}^{(s)}) - (\boldsymbol{T}_2 - \rho_1 \boldsymbol{I})\, \boldsymbol{\Delta u}^{(s)}], \quad s \geq 0 \quad (25)$$

$$\boldsymbol{\Delta u}^{(s+1)} = (\boldsymbol{T}_2 + \rho_2 \boldsymbol{I})^{-1}[-\boldsymbol{\phi}(\boldsymbol{u}^{(s)}) - (\boldsymbol{T}_1 - \rho_2 \boldsymbol{I})\, \boldsymbol{\Delta v}], \quad s \geq 0 \quad (26)$$

Finally, by the assistance of (4.25)- (4.26), from (23) one can secure numerical solution at $(s+1)$ th iteration. For intermingling, it is required that one pick $u^{(0)}$"close" to solution u.

**Results and Conclusions**

In this segment, one have handled the accompanying with two benchmark issues on both uniform & non-uniform work whose right arrangement are known to us and differentiated the results and the relating successive over relaxation and Newton-SOR method. For uniform work, one take $\sigma_k = \sigma = 1$ 7 for non-uniform mesh, one take $\sigma_k = \sigma = a$ constant$\neq 1, k = 1(1)N+1$. By then estimation of first mesh isolating on left is given by

$h_1 = \frac{(1-\sigma)}{(1-\sigma^{N+1})}, \sigma \neq 1$ (27)

Thusly, for non-uniform mesh, given the assessment of $N$, we can figure initial step length $h_1$from above associated and rest of the work is likewise controlled by $\mathrm{h}_{k+1} = \sigma h_k, k = 1(1)N$ . Such a constringent isn't relevant to uniform work case. By taking $\sigma_k = \sigma = 1$, one can apply the strategies clearly to uniform mesh. All calculations were finished utilizing MATLAB. In all cases, one have considered $u^{(0)} = 0$ and the iterations were stopped when the preeminent blunder$|u^{(s+1)} - u^{(s)}| \leq 10^{-10}$resistance was refined. Relationships of plots of right and mathematical answers for picked boundaries for all of the issues analyzed in points of reference are given

**Example**: (*Linear Singular Problem*).

 

$$u'' + \frac{\alpha}{x}u' - \frac{\alpha}{x^2}u = 12x^2 + 4\left[\alpha x^2 + 4x^6 - \frac{\alpha}{x^2}\right]e^{x^4} + 48\int_0^1\left[s^3x^6e^{x^4s^4}\right]ds, \quad (28)$$

$0 < x < 1, \quad 0 < s < 1$

The limit esteems are given by $(0) = 1, u(1) = e$.

The right arrangement of problem is given by $(x) = e^{x^4}$ . The root mean square (RMS) mistakes, optimal estimations $(\rho_{opt}, \rho_{1opt}, \rho_{2opt})$ of relaxation parameters and number of iterations both for TAGE & SOR methods for non-uniform& uniform mesh case are ordered in Table 1 for$(\alpha\sigma)$.

**Example:** (*Nonlinear Singular Problem*)

$$u'' + \frac{\alpha}{x}u' - \frac{\alpha}{x^2}u = uu' + (2+\alpha+x^2)\cosh x - x^3(x\sinh x + 2\cosh x)\cosh x$$

$$+(4+\alpha)\int_0^1 x^2\cosh(xs)ds, \quad 0<x<1,\ 0<s<1. \quad (29)$$

The boundary estimations are given by $u(0) = 0, u(1) = cosh\ (1)$.

The solution of problem is given by $(x) = x^2 cosh\ x$. The RMS errors, optimal values $(\rho_{opt}, \rho_{1opt}, \rho_{2opt})$ of relaxation parameters & number of iterations both for Newton- TAGE & Newton-SOR methods for non-uniform and uniform mesh case are tabulated in Table 2(A) for $(\alpha, \sigma) = (0, 0.8)$.

**Table 1**

*Example*-1: The RMS Errors (Non uniform mesh case)

$\alpha = 1$ , $\sigma$=1.1

| | SOR | | | TAGE | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\rho_{opt}$ | iter | cputime | $\rho_{1opt}$ | $\rho_{2opt}$ | iter | cputime | RMS errors |
| 11 | 1.496 | 39 | 0.004082 s | 0.566 | 0.569 | 29 | 0.003195 s | 0.1503(-03) |
| 21 | 1.666 | 65 | 0.006457 s | 0.336 | 0.340 | 48 | 0.004028 s | 0.3964(-04) |
| 31 | 1.740 | 81 | 0.008218 s | 0.261 | 0.264 | 61 | 0.006792 s | 0.2694(-04) |
| 41 | 1.766 | 91 | 0.011755 s | 0.201 | 0.206 | 71 | 0.008318 s | 0.2175(-04) |
| 51 | 1.787 | 110 | 0.014496 s | 0.170 | 0.174 | 78 | 0.009766 s | 0.1895(-04) |
| 61 | 1.794 | 128 | 0.019992 s | 0.164 | 0.168 | 86 | 0.010686 s | 0.1715(-04) |
| 71 | 1.805 | 132 | 0.022071 s | 0.154 | 0.158 | 88 | 0.011101 s | 0.1583(-04) |
| 81 | 1.828 | 149 | 0.032575 s | 0.134 | 0.139 | 92 | 0.014352 s | 0.1479(-04) |

**Table 2**

*Example*-2: The RMS Errors (uniform mesh case)

$\alpha$= 1, $\sigma$=1

| | SOR | | | TAGE | | | | |
|---|---|---|---|---|---|---|---|---|
| | $\rho_{opt}$ | iter | cputime | $\rho_{1opt}$ | $\rho_{2opt}$ | Iter | cputime | RMS errors |
| 11 | 1.527 | 39 | 0.004332 s | 0.525 | 0.530 | 31 | 0.002968 s | 0.1220(-03) |
| 15 | 1.621 | 53 | 0.005787 s | 0.400 | 0.405 | 43 | 0.003516 s | 0.5182(-04) |
| 21 | 1.707 | 71 | 0.006912 s | 0.277 | 0.283 | 60 | 0.003960 s | 0.1927(-04) |
| 25 | 1.746 | 83 | 0.007802 s | 0.262 | 0.265 | 67 | 0.004512 s | 0.1132(-04) |
| 31 | 1.789 | 101 | 0.012164 s | 0.205 | 0.210 | 82 | 0.005284 s | 0.5793(-05) |
| 41 | 1.835 | 131 | 0.013998 s | 0.157 | 0.164 | 107 | 0.006848 s | 0.2377(-05) |

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 51 | 1.869 | 161 | 0.027139 s | 0.119 | 0.123 | 137 | 0.007131 s | 0.1171(-05) |
| 63 | 1.888 | 197 | 0.051308 s | 0.103 | 0.106 | 158 | 0.015108 s | 0.5853(-06) |
| 71 | 1.903 | 221 | 0.046796 s | 0.086 | 0.092 | 187 | 0.021915 s | 0.3938(-06) |
| 81 | 1.915 | 251 | 0.059709 s | 0.081 | 0.084 | 203 | 0.032910 s | 0.2536(-06) |

**CONCLUSION:-** We have shown another three point variable work technique for third request exactness reliant on cubic spline guess for the arrangement of second order nonlinear two point limit regard issue with driving limits fit as a fiddle. Nevertheless, for $\sigma_k = 1$ it is watched that the proposed strategy reduces to a predictable work technique for exactness of order four. The proposed strategy is suitable when the amount of inward grid reasons for the arrangement space is odd. The strategy is viably associated with both direct and nonlinear issues with solitary coefficients and with driving limits in vital casing. The utilization of TAGE and Newton-TAGE strategies attest the predominance over the contrasting SOR and Newton-SOR techniques as far as number of cycles. The proposed procedures may be extendible to multi-dimensional cases.

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