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Generalized of Ekeland's Variational Principle with Connections with Fixed Point Theory

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ABSTRACT
In this paper, a few augmentations of the Ekeland variational guideline in metric spaces are given for a summed-up pseudo separation. As an application we acquire Caristi's fixed point hypothesis. At that point, by utilizing this outcome, we set up some fixed point hypotheses for set-esteemed contractive mappings.

1. Introduction

We see that a useful restricted under 24 acknowledge its infimum if it has some sort of progression in a geography that renders (nearby) conservativeness to the space of said useful. Anyway in various circumstances of enthusiasm for applications this isn't the situation. For example, functionals characterized in (unbounded dimensional) Hilbert spaces which are nonstop in the standard geography yet not in the weak geography. Issues with this set up can be taken care of proficiently by Ekeland Variational Principle. This rule found in 1972 has found a huge number of uses in various fields of Analysis. It has furthermore served to give direct and rich confirmations of known outcomes. Also, as we see it is a device that unites various outcomes where the underlining thought is a type of estimation. Our inspiration to compose these notes is to make an endeavor to display all of these highlights, which we find numerically very intriguing.Suzuki (2006) in this paper, we consider the strong Ekeland variational principle due to Georgiev. We examine it for functions defined on Banach spaces and on compact metric spaces. Takahashi et al. (2013) in this paper, using the concept of w-distances on a metric space, we first demonstrate a generalized fixed point theorem for mappings without continuity in a complete metric space.Tanaka, Yasuhito (2011) it is often demonstrated that Brouwer's fixed-point theorem cannot be constructively or computably demonstrated. Therefore, Tychonoff's and Schauder's fixed-point theorems likewise cannot be constructively demonstrated. On the other hand, however, Sperner's lemma, which is utilized to demonstrate Brouwer's theorem, can be constructively demonstrated.Torquato, Salvatore (2002) for random media of arbitrary microstructure, exact analytical solutions of the effective properties are unattainable, and so any thorough statement about the effective properties must be in the form of thorough limits. Turinici, Mihai. (2010); Vujanovic, et al. (2004); Wu, Zili. (2003). Have discussed the equivalent formulations of Ekeland's variational principle. Xu, et al. (2006) defined the fixed point index for pitifully inward mappings, investigated its properties and studied the fixed points for such mappings. In this paper, following S. Hu and Y. Sun, we continue to investigate limit conditions, under which the fixed point index for the completely continuous and pitifully inward mapping, denoted by i(A,Ω,P), is equivalent to 1 or 0.

Theorem 1: (Ekeland Principle - feeble form). Let (X, d) be a complete metric space. Let Φ: X → R ∪ {+∞} be lower semi-continuous and limited underneath. Then given any ∈> 0 there exists u∈∈ X with the end goal that

$$\Phi(u_\epsilon) \le \mathrm{Inf}_X \Phi + \epsilon, \quad (1)$$

And

$$\Phi(u_\epsilon) < \Phi(u) + \epsilon d(u, u_\epsilon), \quad \forall u \in X \quad with \quad u \neq u_\epsilon. \quad (2)$$

For future applications one needs a stronger version of Theorem. See that underneath gives information on the whereabouts of the point u_λ. As we will find in Theorem 1 the point u_λ in Theorem is a sort of "almost" critical point henceforth its importance.

Theorem 2: (Ekeland Principle - strong form). Let X is a complete metric space and Φ: X → R ∪ {+∞} a lower semi-continuous function which is limited underneath. Let ∈> 0 and u ∈ X be given with the end goal that

$$\Phi(\overline{u}) \le \mathrm{Inf}_X \Phi + \frac{\epsilon}{2}. \quad (3)$$

Then given λ > 0 there exists uλ   X such that

$$\Phi(u_\lambda) \le \Phi(\overline{u}) \quad (4)$$

$$d(u_\lambda, \overline{u}) \le \lambda \quad (5)$$

$$\Phi(u_\lambda), \Phi(u) + \frac{\epsilon}{\lambda} d(u, u_\lambda) \quad \forall u \neq u_\lambda. \quad (6)$$

Proof: For notational simplification let us put dλ(x, y) = (1/λ)d(x, y). Let us define a partial order in X by

$$u \leq v \Longleftrightarrow \Phi(u) \leq \Phi(v) - \epsilon d_\lambda(u, v).$$

It is straightforward that: (i) (reflexivity) u ≤ u; (ii) (anti-symmetry) u ≤ v and v ≤ u infer u = v; (iii) (transitivity) u ≤ v and v ≤ w suggest u ≤ w; all these three properties for all u, v, ω in X. Presently we define a grouping (S n) of subsets of X as follows. Start with u1 = u and define

$$S_1 = \{u \in X : u \leq u_1\}; \quad u_2 \in S_1 \quad s.t. \quad \Phi(u_2) \leq \text{Inf}_{S_1} \Phi + \frac{\epsilon}{2^2}$$

And inductively

$$S_n = \{u \in X : u \leq u_n\}; \quad u_{n+1} \in S_n \quad s.t. \quad \Phi(u_{n+1}) \leq \text{Inf}_{S_n} \Phi + \frac{\epsilon}{2^{n+1}}.$$

Plainly S1 ⊃ S2 ⊃ S3 ⊃ Each S n is shut: let xj ∊ S n with xj → x ∊ X. We have Φ(xj) ≤ Φ(un) − ∊dλ(xj , un). Taking limits using the lower semi-continuity of Φ and the continuity of d we conclude that x ∊ S n. presently we demonstrate that the diameters of these sets go to zero: diam S n → 0. Indeed, take an arbitrary point x ∊ Sn. On one hand, x ≤ un infers

$$\Phi(x) \leq \Phi(u_n) - \epsilon d_\lambda(x, u_n). \quad (7)$$

On the other hand, we observe that x belongs also to S_{n-1}. So, it is one of the points which entered in the competition when we picked u_n.
So,

$$\Phi(u_n) \leq \Phi(x) + \frac{\epsilon}{2^n} \quad (8)$$

From (7) and (8) we get

$$d_\lambda(x, u_n) \leq 2^{-n} \quad \forall x \in S_n$$

Which gives diam S_n ≤ 2^{-n+1} now we claim that the unique point in the intersection of the S_n's satisfies above conditions. Let then $\bigcap_{n=1}^{\infty} S_n = \{u_\lambda\}$. Since u_λ Ekeland Principle - strong form S1, is clear. Now let u, u_λ. We cannot have u uλ, because otherwise u would belong to the intersection of the S_n's. So u uλ, which means that

$$\Phi(u) > \Phi(u_\lambda) - \epsilon d_\lambda(u, u_\lambda)$$

Thus proving (3) finally to prove (4) we write

$$d_\lambda(\overline{u}, u_n) \leq \sum_{j=1}^{n-1} d_\lambda(u_j, u_{j+1}) \leq \sum_{j=1}^{n-1} 2^{-j}$$

And take limits as n → ∞.
The above results and further theorems in this chapter are due to Ekeland.

Connections with Fixed Point Theory

Presently we show that Ekeland's Principle infers a Fixed Point Theorem due to Caristi. Truly, the two results are equivalent as in Ekeland's Principle can likewise be demonstrated from Caristi's theorem.

Theorem 3: (Caristi Fixed Point Theorem). Let X is a complete metric space, and Φ: X → R∪ {+∞} a lower semi-continuous functional which is limited beneath. Let T: X → 2^X be a multivalued mapping with the end goal that

$$\Phi(y) \leq \Phi(x) - d(x, y), \quad \forall x \in X, \quad \forall y \in Tx. \quad (9)$$

Then there exists x0 X such that x0 T x0.

Proof: Using Theorem 2 with = 1 we find x0 X such that

$$\Phi(x_0) < \Phi(x) + d(x, x_0) \quad \forall x \neq x_0. \quad (10)$$

Now we claim that x0 T x0. Otherwise all y T x0 are such that y x0. So we have from (7) and (8) that

$$\Phi(y) \leq \Phi(x_0) - d(x_0, y) \quad \text{and} \quad \Phi(x_0) < \Phi(y) + d(x_0, y)$$

This cannot hold simultaneously

Proof of Theorem 1 from Theorem 2 Let us utilize the notation d1 = ∈d, which is an equivalent distance in X. Assume by contradiction that there is no u∈ satisfying (2). So for every x ∈ X the set {y ∈ X: Φ(x) ≥ Φ(y) + d1(x, y); y, x} isn't empty. Let us denote this set by Tx. In this manner we have delivered a multi valued mapping T in (X, d1) which satisfies condition (7). By Theorem 2 it should exist x0 ∈ X with the end goal that x0 ∈ T x0. But this is incomprehensible: from the very definition of T x, we have that x T x.

Remark: On the off chance that T is a contraction in a complete metric space, that is, if there exists a constant k, 0 ≤ k < 1, with the end goal that

$d(T_x, T_y) \le kd(x, y), \forall x, y \in X,$

Then T satisfies condition (7) with

$$\Phi(x) = \frac{1}{1-k} d(x, Tx).$$

So part of the Contraction Mapping Principle which says about the existence of a fixed point can be obtained from Theorem 2. Obviously the Contraction Mapping Principle is substantially more than this. Its notable proof uses an iteration system (the method of progressive approximations) which gives an effective computation of the fixed point, with a blunder estimate, etc.

Application of Theorem 1 to Functionals Defined in Banach Spaces, Presently we put more structure on the space X where the functionals are defined. In fact we guess that X is a Banach space. This will permit us to utilize a Differential Calculus, and then we could appreciate better the meaning of the relation (3). Freely speaking (4) has to do with some Newton quotient being little.

Theorem 4: Let X be a Banach space and Φ: X → R a lower semi-continuous functional which is limited beneath. In addition, assume that Φ is Gteaux differentiable at each point x ∈ X. Then for each ∈> 0 there exists u∈∈ X with the end goal that

$$\Phi(u_\epsilon) \le \mathrm{Inf}_X \Phi + \epsilon \quad (11)$$

$$\|D\Phi(u_\epsilon)\|_{X^*} \le \epsilon. \quad (12)$$

Proof: It follows from Theorem 2 that there exists u∈ Ekeland Principle - strong form X such that (1) holds and

$$\Phi(u_\epsilon) \le \Phi(u) + \epsilon\|u - u_\epsilon\| \quad \forall u \in X. \quad (13)$$

Let v ∈ X and t > 0 be arbitrary. Putting u = u∈ + tv in (13) we obtain

$$t^{-1}[\Phi(u_\epsilon) - \Phi(u_\epsilon + tv)] \le \epsilon\|v\|.$$

Passing to the limit as t → 0 we get −hDΦ(u), vi ≤ ∈||v|| for each given v Ekeland Principle - strong form X. Since this inequality is true for v and −v we obtain |h

DΦ(u), vi| ≤ ∈||v||, for all v Ekeland Principle - strong form X. But then

$$\|D\Phi(u)\|_{X^*} = \sup_{v \in V\ v \neq 0} \frac{\langle D\Phi(u), v\rangle}{\|v\|} \le \epsilon.$$

Remark 1: The fact that Φ is Gteaux differentiable does not imply that Φ is lower semi-continuous. One has simple examples, even for $X = R^2$.

Remark 2: In terms of functional equations Theorem 2 methods the following assume that T: X → X* is an operator which is a gradient, i.e., there exists a functional Φ: X → R with the end goal that T = Dφ. The functional Φ is known as the potential of T. On the off chance that Φ satisfies the conditions of Theorem 3, then that theorem says that the equation Tx = x* has a solution x for some x∗ in a chunk of span ∈ around 0 in X*. And this for all ∈> 0 in actuality one could state more if additional conditions are set on Φ.

Theorem 5: In addition to the hypotheses of Theorem 4 accept that there are constants k > 0 and C with the end goal that

$$\Phi(u) \ge k\|u\| - C.$$

Let B* denote the unit ball about the origin in X*. Then DΦ(X) is thick in kB∗.

Proof: We should demonstrate that given ∈> 0 and u∗∈ kB∗ there exists u∈∈ X with the end goal that ||DΦ(u∈) − u∗ ||X∗ ≤ ∈. So, consider the functional Ψ(u) = Φ(u) − h_u∗29, u_i. It is anything but difficult to see that Ψ is lower semi-continuous and Gteaux differentiable. Boundedness beneath follows from (?) so by Theorem 4 we obtain u∈ with the end goal that ||DΨ(u∈) ||X∗ ≤ ∈. Since Dψ(u) = Dψ(u) − u∗, the result follows.

Corollary 1: In addition to the hypotheses of Theorem 4 accept that there exists a continuous function ϕ: [0, ∞) → R with the end goal that ϕ(t)/t → +∞ as t → +∞ and Φ(u) ≥ ϕ(||u||) for all u ∈ X. Then DΦ(X) is thick in X∗.

Proof: Let k > 0. Pick t_0> 0 with the end goal that ϕ(t)/t ≥ k for t > t_0. So Φ(u) ≥ k||u|| if ||u|| > t_0. On the off chance that ||u|| ≤ t_0, Φ(u) ≥ C where C = min{ϕ(t): 0 ≤ t ≤ t_0}. Applying Theorem 4 we see that DΦ(X) is thick in kB∗. Since k is arbitrary the result follows.

For the next result one needs an exceptionally valuable concept, a sort of compactness condition for a functional Φ. We state that a C_1 functional satisfies the Palais - Smale condition [or (PS) condition, for short] if each grouping (un) in X which satisfies

$$|\Phi(u_n)| \le \text{const. and } \Phi'(u_n) \to 0 \quad \text{in} \quad X^*$$

Possesses a convergent (in the norm) subsequence

Theorem 6: Let X be a Banach space and Φ: X → R a C1 functional which satisfies the (PS) condition. Assume in addition that Φ is limited beneath. Then the infimum of Φ is accomplished at a point $u_0 \in X$ and u0 is a critical point of Φ, i.e., Φ′ (u0) = 0.

Proof: Using Theorem 5 we see that for every positive integer n there is un ∈ X with the end goal that

$$\Phi(u_n) \leq \mathrm{Inf}_X \Phi + \frac{1}{n} \quad \|\Phi'(u_n)\| \leq \frac{1}{n}. \quad (14)$$

Using (PS) we have a subsequence (u_nj) and an element $u_0 \in X$ such that u_nj → u_0. Finally from the continuity of both Φ and Φ′ we get (14).

$$\Phi(u_0) = \mathrm{Inf}_X \Phi \quad \Phi'(u_0) = 0 \quad (15)$$

Conclusion:

- Actually the result is true without the continuity of Φ′. The simple existence of the Fr'echet differential at each point does the trick. Indeed, we have only to show that the first statement in (14) infers the second. This is a standard fact in the calculus of Variations. Here it goes its basic proof: take v ∈ X, ||v|| = 1, arbitrary and t > 0. So

$$\Phi(u_0) \leq \Phi(u_0 + tv) = \Phi(u_0) + t\langle \Phi'(u_0), v\rangle + o(t)$$

From which follows that

$$\|\Phi'(u_0)\|_{X^*} = \sup_{\|v\|=1} \langle \Phi'(u_0), v\rangle \leq \frac{o(t)}{t}$$

For all t > 0 making t → 0 we get the result.

- The boundedness beneath of Φ it could be obtained by a condition like the one in Corollary 1. See that a condition like Φ(u) → +∞ promotion ||u|| → ∞ (ordinarily called coerciveness) [or even the stronger one Φ(u)/||u|| → +∞ as ||u|| → +∞] doesn't guarantee that Φ is limited beneath.
- Theorem 5 shows up in Chang with a different proof and restricted to Hilbert spaces. Potentially that proof could be extended to the case of general Banach using a stream given by sub-gradient, as in, instead of the gradient stream.

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