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Mathematical Model In Existence and Uniqueness In Nonlinear Order

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ABSTRACT

This paper is devoted to proving the existence, uniqueness and smoothness of solutions for differential equations with Riemann-Liouville sequential fractional derivatives admitting only the existence of a lower solution. The appropriate fractional Sobolev spaces equipped with a suitable partial order are introduced, and applying some fixed point theorems the results are proved. The approximate solutions are presented to the corresponding initial value problems. A few illustrative examples are given.

1. Introduction

Differential equations occur in the modeling of dynamical behavior of physical processes. If the model of physical system in some ways possesses a memory and hereditary properties, for instance, viscoelastic deformation, anomalous diffusion, stock market, bacterial chemotaxis and complex networks; the corresponding models can be described by the fractional differential equations. Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of biology, chemistery, physics, and so on, and they are gaining much importance and attention, due to their applications. Recently, there are some papers dealing with the existence of solutions for nonlinear fractional differential equation using various methods (fixed point theorems, upper and lower solutions, variational methods, etc.). However, there are few papers which have considered the sequential fractional differential equations. A general theory for linear sequential fractional differential equations with Riemann-Liouville and Caputo derivatives has been presented in [2, 9].

The purpose of this paper is to investigate the solvability of the following fractional initial value problem

$$\mathcal{D}^{\sigma_n} y(x) = f(x, y(x), \mathcal{D}^{\sigma_1} y, \mathcal{D}^{\sigma_2} y, \cdots, \mathcal{D}^{\sigma_{n-1}} y), \qquad (1)$$

$$\mathcal{D}^{\sigma_k - 1} y(x)\,|_{x=0} = b_k, \quad (k = 1, \cdots, n), \qquad (2)$$

Where,

$$\mathcal{D}^{\sigma_k} \equiv D^{\alpha_k} D^{\alpha_{k-1}} \cdots D^{\alpha_1}, \qquad \mathcal{D}^{\sigma_k - 1} \equiv I^{1-\alpha_k} D^{\alpha_{k-1}} \cdots D^{\alpha_1}, \qquad (3)$$

$$\sigma_0 = 0,\ \sigma_k = \sum_{j=1}^{k} \alpha_j, \quad (k = 1, 2, \cdots, n), \qquad 0 < \alpha_j < 1, \quad (j = 1, 2, \cdots, n), \qquad (4)$$

And Dα is the classical Riemann-Liouville fractional derivative of order α. The notation Dσ was introduced in [9] for sequential fractional derivative.

Problem (1)-(2) is of interest because it appears in mathematical models of physical phenomena. For example, when n = 2 we get the steady nonlinear fractional advection-dispersion equation [8, 9, 15, 16]. For n = 4 and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \frac{1}{2}$ we obtain a general case of Bagley-Torvik equation with sequential fractional derivatives which arises, for example, in the modeling of the motion of a rigid plate immersed in a Newtonian fluid. Another example for an application of problem (1)-(2) is the Basset equation which describes the forces that occur when a spherical object sinks in a incompressible viscous fluid [5, 6].

2. Preliminaries

In this section we state some definitions and lemmas which are crucial in our analysis. Here and in the sequel, we assume p ≥ 1 and q is the conjugate exponent of p; that is, $\frac{1}{p} + \frac{1}{q} = 1.$ Let T > 0. In what follows, Γ denotes the Gamma function.

Definition 1. The Riemann-Liouville fractional integral I α of order α ≥ 0 of a function y : [0, T] → R is defined by

$$I^{\alpha} y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)\,dt,$$

provided the right-hand side is defined for almost everywhere x ∈ [0, T]. We note that for y ∈ L 1 [0, T] we have that I αy ∈ L 1 [0, T]. For α = 0, we set I α := I, the identity operator.

Definition 2. The Riemann-Liouville fractional derivative Dα of order 0 < α < 1 of a function y : [0, T] → R is defined by

$$D^{\alpha}y(x) = \frac{d}{dx}I^{1-\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}y(t)dt,$$

provided the right-hand side is defined for almost everywhere x ∈ [0, T].

Definition 3. Let 0 < α < 1. A function y ∈ L 1 [0, T] is said to have a summable fractional derivative $D^{\alpha}y$, if $I^{1-\alpha}y \in AC[0,T]$ where AC[0, T] represents the space of absolutely continuous functions on [0, T].

Proposition 1. The fractional integration operator I α with α > 0 is bounded in L p [0, T],

$$\|I^{\alpha}y\|_{L^p} \le \frac{T^{\alpha}}{\Gamma(\alpha+1)}\|y\|_{L^p}.$$

Lemma 1. Let $0 < \frac{1}{p} < \alpha < 1$ and $y \in L^p[0,T]$, then $I^{\alpha}y$ is continuous and $\lim_{x\to 0^+} I^{\alpha}y(x) = 0.$ Consequently, I αy can be continuously extended by 0 in x = 0.

Proposition 2. Let $\alpha, \beta \ge 0$ and $p \ge 1$. If $y \in L^p[0,T]$, then

$$I^{\alpha}I^{\beta}y(x) = I^{\alpha+\beta}y(x),$$

almost everywhere on [0, T].

Proposition 3 Let $0 < \alpha < 1$ and $p \ge 1$. If $y \in L^p[0,T]$, then

$$D^{\alpha}I^{\alpha}y(x) = y(x),$$

almost everywhere on [0, T].

Proposition 4 Let $0 < \alpha < 1$ and $p \ge 1$. Let $y, D^{\alpha}y \in L^p[0,T]$, then

$$I^{\alpha}D^{\alpha}y(x) = y(x) - \frac{I^{1-\alpha}y(0)}{\Gamma(\alpha)}x^{\alpha-1},$$

almost everywhere on [0, T].

Now we present the fixed point theorems which play main role in our discussion.

Definition 4. If (X, ≼) is a partially ordered set and S : X → X, we say that S is nondecreasing if x ≼ y implies S(x) ≼ S(y).

Theorem 1.(Partially Fixed Point Theorem). Let (X, ≼) be a partially ordered set and let d be a metric on X such that (X, d) is a complete metric space. Furthermore, let S : X → X be a continuous and non-decreasing mapping such that

$$\exists\, 0 \le k < 1 : d(S(x), S(y)) \le kd(x,y), \quad \forall\, y \preceq x,$$
$$\exists x_0 \in X : x_0 \preceq S(x_0).$$

Then S has a fixed point.

Theorem 2. Assume the hypotheses of Theorem 1, except for the continuity of S. Moreover, we assume that for a non-decreasing sequence xn → x in X, there exists a subsequence {xnk } of {xn} such that every term is comparable to x. Then S has a fixed point.

Theorem 3. Let all the conditions of Theorem 1 (resp. Theorem 2) be fulfilled and let the following condition holds:

For every x, y ∈ X, there exists z ∈ X which is comparable to x and y.

Then S has a unique fixed point ¯x. Moreover, for every x ∈ X,

$$\lim_{n\to\infty} S^n(x) = \bar{x}.$$

3. Partially ordered fractional Sobolev spaces

Hereafter we suppose αk, σk and Dσk are as in (3)-(4). Now we introduce the following fractional Sobolev spaces and equip them with a partially order.

Definition 5. We define the following spaces of functions

$$\mathbb{E}^{\sigma_n,p}[0,1] := \{y \in L^p[0,1] : \mathcal{D}^{\sigma_k}y \in L^p[0,1], \quad k = 1,\cdots,n\},$$

And,

$$\mathbb{E}_0^{\sigma_n,p}[0,1] := \{y \in L^p[0,1] : \mathcal{D}^{\sigma_k}y \in L^p[0,1], \quad \mathcal{D}^{\sigma_k-1}y(x)\,|_{x=0} = 0, \quad k = 1,\cdots,n\},$$

endowed with the norm

$$\|y\|_{\mathbb{E}^{\sigma_n,p}} = \sum_{k=0}^{n} \|\mathcal{D}^{\sigma_k} y\|_{L^p}.$$

We notice that here Dσk y ∈ L p [0, 1] means that Dσk−1y ∈AC[0, 1], and its derivative which exists almost everywhere on [0, 1] belongs to L p [0, 1].

Theorem 4. The space $\mathbb{E}_0^{\sigma_n,p}[0,1]$ is a Banach space for 1 ≤ p ≤ ∞.

Proof. Let $\{y_j\}$ be a Cauchy sequence in $\mathbb{E}_0^{\sigma_n,p}[0,1]$, then $\{y_j\}$ and $\{\mathcal{D}^{\sigma_k} y_j\}_{k=1}^n$ are Cauchy sequences in $L^p[0,1]$. It follows that

$$y_j \xrightarrow{L^p} y, \quad \mathcal{D}^{\sigma_k} y_j \xrightarrow{L^p} y^{(\sigma_k)}, \quad k = 1, \cdots, n.$$

As $\{\mathcal{D}^{\sigma_k} y_j\}_{k=1}^n \subseteq L^p$, using Proposition 4 we deduce that for any $j \in \mathbb{N}$ and $k = 1, \cdots, n$,

$$I^{\alpha_k} \mathcal{D}^{\sigma_k} y_j(x) = \mathcal{D}^{\sigma_{k-1}} y_j(x) - \frac{\mathcal{D}^{\sigma_k -1} y(x)\mid_{x=0}}{\Gamma(\alpha_k)} x^{\alpha_k - 1},$$

therefore, for any $j \in \mathbb{N}$, we have

$$\begin{cases} y_j(x) = I^{\alpha_1} \mathcal{D}^{\sigma_1} y_j(x), \\ \mathcal{D}^{\sigma_1} y_j(x) = I^{\alpha_2} \mathcal{D}^{\sigma_2} y_j(x), \\ \vdots \\ \mathcal{D}^{\sigma_{n-1}} y_j(x) = I^{\alpha_n} \mathcal{D}^{\sigma_n} y_j(x). \end{cases} \tag{5}$$

In view of (5) and the continuity of Riemann-Liouville fractional integral operator from L p to L p, we deduce

$$\mathcal{D}^{\sigma_{k-1}} y_j(x) \xrightarrow{L^p} I^{\alpha_k} y^{(\sigma_k)}, \quad k = 1, 2, \cdots, n.$$

Therefore, we have

$$y^{(\sigma_{k-1})} = I^{\alpha_k} y^{(\sigma_k)}, \quad k = 1, 2, \cdots, n,$$

almost everywhere on [0, 1]. Finally, using Proposition 3, we immediately get that for k = $1, \cdots, n$, $y^{(\sigma_k)} = \mathcal{D}^{\sigma_k} y$, and hence the result.

Definition 6. We define the following order relation for E σn,p[0, 1],

$$y, \tilde{y} \in \mathbb{E}^{\sigma_n,p}[0,1], \quad y \preceq \tilde{y} \iff \mathcal{D}^{\sigma_k} y(x) \le \mathcal{D}^{\sigma_k} \tilde{y}(x), \quad \text{a.e. } x \in [0,1], \quad k = 0, 1, \cdots, n.$$

Lemma 2. $\mathbb{E}_0^{\sigma_n,p}[0,1]$ is a partially ordered set and every pair of elements has a lower bound and an upper bound.

Proof. It is easy to see that $\mathbb{E}_0^{\sigma_n,p}[0,1]$ is a partially ordered set. Now we prove that every pair of elements in $\mathbb{E}_0^{\sigma_n,p}[0,1]$ has a lower bound and an upper bound. Let $y, \tilde{y} \in \mathbb{E}_0^{\sigma_n,p}[0,1]$ and define $\underline{z}(x) = I^{\sigma_n} \min\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\}$ and $\overline{z}(x) = I^{\sigma_n} \max\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\}$. Then from Propositions 1 and 3, we have $\{\mathcal{D}^{\sigma_k} \underline{z}(x)\}_{k=0}^n \subseteq L_p[0,1]$ and $\{\mathcal{D}^{\sigma_k} \overline{z}(x)\}_{k=0}^n \subseteq L_p[0,1]$. On the other hand, from Propositions 2 and 3, we get

$$\mathcal{D}^{\sigma_k -1} \underline{z}(x) = I^1 I^{\sigma_n - \sigma_k} \min\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\},$$

and

$$\mathcal{D}^{\sigma_k -1} \overline{z}(x) = I^1 I^{\sigma_n - \sigma_k} \max\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\},$$

for $k = 1, \cdots, n$. This shows that $\mathcal{D}^{\sigma_k -1} \underline{z}(x)(x)\mid_{x=0} = 0$ and $\mathcal{D}^{\sigma_k -1} \overline{z}(x)(x)\mid_{x=0} = 0$ for $k = 1, \cdots, n$. Thus $\underline{z}, \overline{z} \in \mathbb{E}_0^{\sigma_n,p}[0,1]$. Finally, from Proposition 3, we have

$$\mathcal{D}^{\sigma_n} \underline{z}(x) = \min\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\},$$

and

$$\mathcal{D}^{\sigma_n} \overline{z}(x) = \max\{\mathcal{D}^{\sigma_n} y(x), \mathcal{D}^{\sigma_n} \tilde{y}(x)\},$$

almost everywhere on $[0,1]$. Therefore, we get

$$\mathcal{D}^{\sigma_n} \underline{z}(x) \le \mathcal{D}^{\sigma_n} y(x), \qquad \mathcal{D}^{\sigma_n} \underline{z}(x) \le \mathcal{D}^{\sigma_n} \tilde{y}(x), \tag{6}$$

and

$$\mathcal{D}^{\sigma_n}\overline{z}(x) \geq \mathcal{D}^{\sigma_n}y(x), \qquad \mathcal{D}^{\sigma_n}\overline{z}(x) \geq \mathcal{D}^{\sigma_n}\tilde{y}(x). \tag{7}$$

Now since Riemann-Liouville fractional integral I^{σ_n} is a monotone operator, we apply the fractional integral I^{σ_n} on both sides of inequalities (6) and (7), and by Proposition 4, we have

$$\mathcal{D}^{\sigma_{n-1}}\underline{z}(x) \leq \mathcal{D}^{\sigma_{n-1}}y(x), \qquad \mathcal{D}^{\sigma_{n-1}}\underline{z}(x) \leq \mathcal{D}^{\sigma_{n-1}}\tilde{y}(x),$$

and

$$\mathcal{D}^{\sigma_{n-1}}\overline{z}(x) \geq \mathcal{D}^{\sigma_{n-1}}y(x), \qquad \mathcal{D}^{\sigma_{n-1}}\overline{z}(x) \geq \mathcal{D}^{\sigma_{n-1}}\tilde{y}(x).$$

Lemma 3. Every non-decreasing sequence $\{y_j\} \to y$ in $\mathbb{E}_0^{\sigma_n,p}[0,1]$ has a convergent subsequence such that every term is comparable to y.

Proof. Let $\{y_j\} \subseteq \mathbb{E}_0^{\sigma_n,p}[0,1]$ be a non-decreasing sequence such that $y_j \to y$. Then, $\mathcal{D}^{\sigma_k}y_j \to \mathcal{D}^{\sigma_k}y$ in $L^p[0,1]$ for $k = 0, 1, \cdots, n$. Therefore, there exists a subsequence (still denoted by $\{y_j\}$) so that

$$\begin{cases} y_j \to y, & \text{a.e. in } [0,1], \\ \mathcal{D}^{\sigma_1}y_j \to \mathcal{D}^{\sigma_1}y, & \text{in } L_p[0,1], \\ \vdots & \\ \mathcal{D}^{\sigma_n}y_j \to \mathcal{D}^{\sigma_n}y, & \text{in } L_p[0,1]. \end{cases}$$

By repeating this process n times, we get a subsequence (still denoted by $\{y_j\}$) so that

$$\begin{cases} y_j \to y, & \text{a.e. in } [0,1], \\ \mathcal{D}^{\sigma_1}y_j \to \mathcal{D}^{\sigma_1}y, & \text{a.e. in } [0,1], \\ \vdots & \\ \mathcal{D}^{\sigma_n}y_j \to \mathcal{D}^{\sigma_n}y, & \text{a.e. in } [0,1]. \end{cases} \tag{8}$$

Therefore, since the original sequence is non-decreasing, the relation (8) shows that for every $j \in \mathbb{N}$, $y_j \preceq y$.

4. Existence and uniqueness

In this section, we intend to give existence and uniqueness results for the initial value problem (1)-(2).

Definition 7. A function y0 ∈ E σn,p[0, T] is called a lower solution of the initial value problem (1)-(2), if it satisfies the initial conditions (2) and

$$\mathcal{D}^{\sigma_n}y_0(x) \leq f(x, y_0(x), \mathcal{D}^{\sigma_1}y_0(x), \mathcal{D}^{\sigma_2}y_0(x), \cdots, \mathcal{D}^{\sigma_{n-1}}y_0(x)),$$

almost everywhere on [0, 1].

To prove the main results, we need the following assumptions:

(H1) $f : [0,1] \times \mathbb{R}^n \to \mathbb{R}$ be a function such that $f(x, y(x), \mathcal{D}^{\sigma_1}y(x), \cdots, \mathcal{D}^{\sigma_{n-1}}y(x)) \in L^p[0,1]$ for every $y \in \mathbb{E}^{\sigma_n,p}[0,1]$.

(H2) f is non-decreasing in all its arguments except for the first argument and there exists $L > 0$ such that

$$f(x, y_1, \cdots, y_n) - f(x, \tilde{y}_1, \cdots, \tilde{y}_n) \leq L\sum_{k=1}^{n}(y_k - \tilde{y}_k), \qquad y_k \geq \tilde{y}_k.$$

For simplicity, we first confine our attention to the following initial value problem

$$\mathcal{D}^{\sigma_n}y(x) = f(x, y(x) + g(x), \mathcal{D}^{\sigma_1}(y(x) + g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y(x) + g(x))), \tag{9}$$

$$\mathcal{D}^{\sigma_k - 1}y(x)\,|_{x=0} = 0, \quad (k = 1, \cdots, n), \tag{10}$$

where g ∈ E σn,p[0, 1] is a given function.

Theorem 5. Assume that (H1)-(H2) hold. Then there exists 0 < T ≤ 1 such that the existence of a lower solution for (9)-(10) in E σn,p[0, T] provides the existence of a unique solution y ∈ E σn,p[0, T] for (9)-(10).

Proof. We choose T > 0 such that the inequality

$$\sum_{k=0}^{n} \frac{LT^{\sigma_n-\sigma_k}}{\Gamma(\sigma_n-\sigma_k+1)} < 1,$$

holds. Now we define $S : \mathbb{E}_0^{\sigma_n,p}[0,T] \to \mathbb{E}_0^{\sigma_n,p}[0,T]$ by

$$Sy(x) = I^{\sigma_n} f(x, y(x)+g(x), \mathcal{D}^{\sigma_1}(y(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y(x)+g(x))),$$

First, by Propositions 2 and 3, it is clear that if $y \in \mathbb{E}_0^{\sigma_n,p}[0,T]$, then

$$\mathcal{D}^{\sigma_k} Sy = I^{\sigma_n-\sigma_k} f, \qquad \mathcal{D}^{\sigma_k-1} Sy = I^1 I^{\sigma_n-\sigma_k} f,$$

for $k = 0, 1, \cdots, n$. This shows that $Sy \in \mathbb{E}_0^{\sigma_n,p}[0,T]$.

Now let $y, \tilde{y} \in \mathbb{E}_0^{\sigma_n,p}[0,1]$ with $y \preceq \tilde{y}$. From the non-decreasing assumption of f in all its arguments except for the first and using the monotonicity of Riemann-Liouville fractional integral operator, for $k = 0, 1, \cdots, n$, we obtain

$$\begin{aligned}
\mathcal{D}^{\sigma_k} Sy(x) &= \mathcal{D}^{\sigma_k} I^{\sigma_n} f(x, y(x)+g(x), \mathcal{D}^{\sigma_1}(y(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y(x)+g(x))) \\
&= I^{\sigma_n-\sigma_k} f(x, y(x)+g(x), \mathcal{D}^{\sigma_1}(y(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y(x)+g(x))) \\
&\le I^{\sigma_n-\sigma_k} f(x, \tilde{y}(x)+g(x), \mathcal{D}^{\sigma_1}(\tilde{y}(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(\tilde{y}(x)+g(x))) \\
&= \mathcal{D}^{\sigma_k} S\tilde{y}(x),
\end{aligned}$$

almost everywhere on [0, T]. This proves that S is a non-decreasing operator. Also, for ˜y ≼ y, we have

$$\begin{aligned}
\|Sy(x) - S\tilde{y}(x)\|_{\mathbb{E}^{\sigma_n,p}} &= \|I^{\sigma_n} f(x, y(x)+g(x), \cdots, \mathcal{D}^{\sigma_{n-1}}(y(x)+g(x))) \\
&\qquad - f(x, \tilde{y}(x)+g(x), \cdots, \mathcal{D}^{\sigma_{n-1}}(\tilde{y}(x)+g(x)))\|_{\mathbb{E}^{\sigma_n,p}} \\
&\le L\|I^{\sigma_n}[(y(x)-\tilde{y}(x)) + \cdots + (\mathcal{D}^{\sigma_{n-1}}y(x) - \mathcal{D}^{\sigma_{n-1}}\tilde{y}(x))]\|_{\mathbb{E}^{\sigma_n,p}} \\
&= L\|I^{\sigma_n}[(y(x)-\tilde{y}(x)) + \cdots + \mathcal{D}^{\sigma_{n-1}}(y(x)-\tilde{y}(x))]\|_{\mathbb{E}^{\sigma_n,p}} \\
&= L\sum_{k=0}^{n} \|I^{\sigma_n-\sigma_k}[(y(x)-\tilde{y}(x)) + \cdots + \mathcal{D}^{\sigma_{n-1}}(y(x)-\tilde{y}(x))]\|_{L^p} \\
&\le L\sum_{k=0}^{n} \frac{T^{\sigma_n-\sigma_k}}{\Gamma(\sigma_n-\sigma_k+1)} \|[(y(x)-\tilde{y}(x)) + \cdots + \mathcal{D}^{\sigma_{n-1}}(y(x)-\tilde{y}(x))]\|_{L^p} \\
&\le L\sum_{k=0}^{n} \frac{T^{\sigma_n-\sigma_k}}{\Gamma(\sigma_n-\sigma_k+1)} \|y(x)-\tilde{y}(x)\|_{L^p} + \cdots + \|\mathcal{D}^{\sigma_{n-1}}(y(x)-\tilde{y}(x))\|_{L^p} \\
&\le L\sum_{k=0}^{n} \frac{T^{\sigma_n-\sigma_k}}{\Gamma(\sigma_n-\sigma_k+1)} \|y(x)-\tilde{y}(x)\|_{\mathbb{E}^{\sigma_n,p}}.
\end{aligned}$$

On the other hand, we have $y_0 \in \mathbb{E}_0^{\sigma_n,p}[0,T]$ such that

$$\mathcal{D}^{\sigma_n} y_0(x) \le f(x, y_0(x)+g(x), \mathcal{D}^{\sigma_1}(y_0(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y_0(x)+g(x))),$$

almost everywhere on $[0,T]$. Therefore, by Proposition 3, we have

$$\begin{aligned}
\mathcal{D}^{\sigma_n} y_0(x) &\le \mathcal{D}^{\sigma_n} I^{\sigma_n} f(x, y_0(x)+g(x), \mathcal{D}^{\sigma_1}(y_0(x)+g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(y_0(x)+g(x))) \\
&= \mathcal{D}^{\sigma_n} Sy_0(x),
\end{aligned}$$

almost everywhere on [0, T]. Since Riemann-Liouville fractional integral operator is a monotone operator, we apply the fractional integral I αn on both sides of inequality (11) and by Proposition 4, we deduce

$$\mathcal{D}^{\sigma_{n-1}} y_0(x) \le \mathcal{D}^{\sigma_{n-1}} Sy_0(x),$$

almost everywhere on [0, T]. We note that Sy0 ∈ E σn,p 0 [0, T]. By repeating this process n times, we deduce y0 ≼ Sy0. Thus an application of the Theorem 3, together with Lemmas 2 and 3, yield the existence and uniqueness of the solution of (9) on E σn,p 0 [0, T]. Moreover, the unique solution of (9)-(10) can be obtained as limn→∞ S n (y) for every y ∈ E σn,p 0 [0, T].

We are now ready to prove the existence result for Problem (1)-(2).

Theorem 6. Let (H1)-(H2) hold. Assume that y0 is a lower solution of (1)-(2) in E σn,1 [0, 1]. Then there exists a unique solution y ∈ E σn,1 [0, T] of (1)-(2) for some 0 < T ≤ 1.

Proof. Let $g(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\sigma_k)} x^{\sigma_k - 1}$ where the coefficients b_k are given as initial conditions (2). From the fact that $D^\alpha x^{\alpha-1} = 0$, we have $\mathcal{D}^{\sigma_n} g(x) = 0$, and utilizing $I^\alpha x^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} x^{\beta+\alpha-1}$ and $D^\alpha x^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} x^{\beta-\alpha-1}$, we get $\mathcal{D}^{\sigma_k-1} g(x)|_{x=0} = b_k$ for every $k = 1, 2, \cdots, n$. Now employing the transformation $z(x) = y(x) - g(x)$ in the problem (1)-(2), we obtain

$$\mathcal{D}^{\sigma_n} z(x) = f(x, z(x) + g(x), \mathcal{D}^{\sigma_1}(z(x) + g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(z(x) + g(x))), \qquad (12)$$
$$\mathcal{D}^{\sigma_k - 1} z(x)\,|_{x=0} = 0, \quad (k = 1, \cdots, n),$$

On the other hand, let $z_0(x) = y_0(x) - g(x)$, where $y_0(x)$ is a lower solution of (1)- (2). Then, we have

$$\begin{aligned} \mathcal{D}^{\sigma_n} z_0(x) &= \mathcal{D}^{\sigma_n} y_0(x) \qquad (13) \\ &\leq f(x, y_0(x), \mathcal{D}^{\sigma_1} y_0(x), \cdots, \mathcal{D}^{\sigma_{n-1}} y_0(x)) \\ &= f(x, z_0(x) + g(x), \mathcal{D}^{\sigma_1}(z_0(x) + g(x)), \cdots, \mathcal{D}^{\sigma_{n-1}}(z_0(x) + g(x))). \end{aligned}$$

Therefore, $z_0(x)$ is a lower solution of (12)-(13). So in view of Theorem 5 there exists a unique solution $\bar{z} \in \mathbb{E}^{\sigma_n,1}[0, T]$ of (12)-(13). Thus $\bar{y}(x) = \bar{z}(x) + g(x)$ is a unique solution of (1)-(2). Moreover, the unique solution $\bar{z} \in \mathbb{E}^{\sigma_n,1}[0, T]$ of (1)-(2) can be obtained as $\lim_{m\to\infty} y_m(x)$ where

$$y_m(x) = \sum_{k=1}^{n} \frac{b_k}{\Gamma(\sigma_k)} x^{\sigma_k - 1} + I^{\sigma_n} f(x, y_{m-1}(x), \mathcal{D}^{\sigma_1} y_{m-1}(x), \cdots, \mathcal{D}^{\sigma_{n-1}} y_{m-1}(x)).$$

5. Conclusion

In this paper, we have presented some results dealing with the existence, uniqueness and smoothness of solutions for nonlinear sequential fractional differential equations. As a first step, we constructed appropriate fractional Sobolev spaces and equipped them with a suitable partial order. The existence and uniqueness of solutions of these equations are obtained using partially fixed point theorems. The advantage of this method arises from the fact that it is a constructive method that yields an analytic approximate solution of problem (1)-(2). Our approach is simple and is applicable to a variety of real world problems. For the illustration of the results, we have considered some examples.

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