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# A Study of Resolve Ordinary Differential Equations of Second Order Using Complex Numbers

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**ABSTRACT**

Ordinary differential equations (ODEs) are a subject with a broad variety and sometimes in both the last high-school and early stages of tertiary education it is important to introduce it to students. Use of complex variable analyzes a subject that students in question may not have previously discussed, among other methods for solving secondary odes with constant coefficients. The aim of this article is to provide an alternative approach to the general solution without using complex numbers for these types of equations.

## 1. Introduction

The need for teaching differential equations and their applications arises—not only in mathematical courses, but also in subjects such as physics, engineering, mechanics, etc.—in the last year of senior high school, as well as in the early stages of higher education. In order for a sixth-former or a non-mathematics major college freshman to successfully attend an introductory differential equations course, he/she must at least have good knowledge of basic differentiation and integration techniques. A student at that level can usually cope with a first-order ordinary differential equation (ODE), which is either variable separable, homogeneous, exact or easily converted into an exact one, by applying the appropriate integrating factor.

$$a\varphi''(x) + b\varphi'(x) + c\varphi(x) = f(x) \quad (1)$$

Nevertheless, it is second-order ODE with constant coefficients that are often regarded as hard to handle by students (Nicholas 1991). If the attempt to overcome this equation is based on complex variables and the use of formulae of Euler (Cheung, 1993), the issue is much greater. It appears to be so due to the lack of nuanced analyzes of variable values by sixth graduates or students in the first year of higher education. However, type ODEs (1), explaining a good number of phenomena in physics and mechanics including the electrical circuit, damp and damp spring vibrations, basic harmonic motion, etc. are found in many other topics, and not just in an introductory course of differential equations.

It is well-known that the homogeneous second-order ODE with constant coefficients

$$a\varphi''(x) + b\varphi'(x) + c\varphi(x) = 0 \quad (2)$$

can be solved by examining the roots of its quadratic auxiliary equation

$$am^2 + bm + c = 0, \quad (3)$$

given by

$$m_{1,2} = \left(-b \pm \sqrt{b^2 - 4ac}\right)/2a. \quad (4)$$

(i) In case b2 – 4ac =0, equation (3) has only one root, i.e. m = –b/2a, of multiplicity two and the differential equation (2), possesses two linearly independent solutions

$$\varphi_1(x) = e^{mx} \text{ and } \varphi_2(x) = xe^{mx}.$$

Its general solution can then be written as

$$\varphi(x) = c_1 e^{mx} + c_2 x e^{mx}, \quad (5)$$

where c1 and c2 are arbitrary constants.

(ii) In case b2 – 4ac =D > 0, equation (3) has two distinct real roots

$$m_1 = \left(-b + \sqrt{\Delta}\right)/2a \text{ and } m_2 = \left(-b - \sqrt{\Delta}\right)/2a.$$

Then the general solution of equation (2) is given by

$$\varphi(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \quad (6)$$

As above, c1 and c2 are arbitrary constants.

(iii) In case b2 – 4ac < 0, a student not having been taught complex numbers, states that the quadratic equation (3) has no real roots and of course does not know how to proceed further.

A method of achieving the general solution (Nicholas 1991) has been proposed for the inhomogenic ODE (1) with the goal of making the weaker students more understandable. The methodology was based on a completion-the-square approach by reversing the normal solution of the equation, i.e. at the end, rather than at start, the roots of the auxiliary equation ( 3). The author of

 

(Cheung, 1993) was thus guided to approach the homogeneous ODE (2) in such a manner that the use of complex numerical analysis was not required in case b2 – 4ac < 0. In recent years, another approach has been demonstrated with no complex numbers in order to derive general solution for homogeneous and inhomogeneous second-order ODEs with consistent coefficients.

The objective of this work is to supplement the aforementioned methods and yet another technique in solving form (1) and (2) second-order ODEs, without using complex numbers. We assume that each teacher who has to deal with sixth grade teachers or college students who have to learn differential equations or a physics professor should have as many methods and means as possible on hand. It is much more important if the students involved have not been taught complex numerical analysis in the past.

Ultimately, one must agree with the great mathematician Sir Michael Atiyah, who recently commented: any good theorem should have many proofs more than average, (Raussen&Skaau, 2005; Scimone, 2008). For two reasons: different research typically has different strengths and limitations and generalizes into different directions — it's not just repeatments.

Following Dobos Fiscal (2007), we continue with the solution to the well-known and most popular differential equation, the simple harmonic movement.

## 2. Solving The Differential Equation Of Simple Harmonic Motion

A mathematical relationship, usually an ordinary or part differential equation, explains most physical phenomena and this is the case of the simple, harmonic, motion of an entity attached to a string and initially motivated. It is the second order ODE, the relation it defines

$$\varphi''(x) + \omega^2\varphi(x) = 0, \quad (7)$$

where ! is a constant and the auxiliary equation m2 + !2 = 0, clearly does not have real roots.

Two first functions, namely '1 = sin! Dobos Tom (2007). x & '2 = cos = cos! (x) that make use of the well-known trigonometry identity sin2 and are linear sally independent solutions of (7)! ximum + cos2! x = 1, the overall solution for the ODE above was obtained.

We here point out that even a sixth man can be easily explained by showing that the determinant known as the Wronskian is not zero, the linear independence of any two functions. The Wronski functions '1 and' 2 above, for example, are defined

$$\begin{vmatrix} \varphi_1 & \varphi_2 \\ \varphi_1' & \varphi_2' \end{vmatrix} = \begin{vmatrix} \sin\omega x & \cos\omega x \\ \cos\omega x & -\sin\omega x \end{vmatrix} = -\sin^2\omega x - \cos^2\omega x = -1$$

We shall begin our approach by seeking a solution in the form of a series, i.e. assume that the infinite series

$$\varphi(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n + \cdots = \sum_{n=0}^{\infty} c_nx^n$$

formally satisfies equation (7), then

$$\varphi'(x) = c_1 + 2c_2x + 3c_3x^2 + \cdots = \sum_{n=1}^{\infty} nc_nx^{n-1},$$

$$\varphi''(x) = 2c_2 + 6c_3x + 12c_4x^2 + \cdots = \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2}.$$

Substituting into (7) we have

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \omega^2\sum_{n=0}^{\infty} c_nx^n = 0.$$

Now we replace n by n – 2 in the second series to obtain

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \omega^2\sum_{n=2}^{\infty} c_{n-2}x^{n-2} = 0$$

Or

$$\sum_{n=2}^{\infty} \left[n(n-1)c_n + \omega^2c_{n-2}\right]x^{n-2} = 0. \quad (8)$$

In order for (8) to be satisfied for every x 2 R and n = 2, 3, ... , the sum of the constant coefficients of all the powers of x must equal zero, and hence

$$c_n = -\frac{\omega^2}{n(n-1)}c_{n-2}, \ (n \geq 2). \quad (9)$$

Relation (9) is called the recursion formula for the coefficients of the series in (8), from which we successively obtain

$$\text{for } n=2,\ c_2=\frac{-\omega^2}{2!}c_0,\ \text{for } n=3,\ c_3=\frac{-\omega^3}{\omega\cdot 3!}c_1,$$

$$\text{for } n=4,\ c_4=\frac{\omega^4}{4!}c_0,\ \text{for } n=5,\ c_5=\frac{\omega^5}{\omega\cdot 5!}c_1,$$

$$\text{for } n=6,\ c_6=-\frac{\omega^6}{6!}c_0,\ \text{for } n=7,\ c_7=-\frac{\omega^7}{\omega\cdot 7!}c_1 \text{ and so on.}$$

Of course, even power coefficients of x are several of c0 and odd power coefficients are many of c1. In addition, c0 and c1 are constants undefined.

We now get the solution to equating (7) in the following form by splitting the sequence of even and odd powers of x.

$$\varphi(x)=c_0\left[1-\frac{(\omega x)^2}{2!}+\frac{(\omega x)^4}{4!}-\frac{(\omega x)^6}{6!}+\cdots\right]+\frac{c_1}{\omega}\left[\omega x-\frac{(\omega x)^3}{3!}+\frac{(\omega x)^5}{5!}-\frac{(\omega x)^7}{7!}+\cdots\right]$$

Or

$$\varphi(x)=c_0\sum_{n=0}^{\infty}(-1)^n\frac{(\omega x)^{2n}}{(2n)!}+\frac{c_1}{\omega}\sum_{n=0}^{\infty}(-1)^n\frac{(\omega x)^{2n+1}}{(2n+1)!}.$$

From elementary calculus, for instance [Anton (2002), p. 694], it is known that

$$\sum_{n=0}^{\infty}(-1)^n\frac{(\omega x)^{2n}}{(2n)!}=\cos\omega x \text{ and } \sum_{n=0}^{\infty}(-1)^n\frac{(\omega x)^{2n+1}}{(2n+1)!}=\sin\omega x.$$

Accordingly, the general solution to equation (7) can now be written as

$$\varphi(x)=\alpha\cos\omega x+\beta\sin\omega x, \tag{10}$$

with a, b arbitrary constants

### 3. Homogeneous Second-Order Odes With Constant Coefficients

In this section, we turn our attention to the second-order homogeneous ODE of the form

$$a\varphi''(x)+b\varphi'(x)+c\varphi(x)=0, \tag{11}$$

where a, b and c are constants such that b2 – 4ac < 0.

$\varphi''(x)+p\varphi'(x)+q\varphi(x)=0,$ (12)(

Letting $p=b/a$ and $q=c/a,\ (a\neq 0),$ equation (11) becomes

$$\varphi''(x)+p\varphi'(x)+q\varphi(x)=0, \tag{12}$$

whose auxiliary equation is

$m^2+pm+q=0.$ (13)

The roots of equation (13) are given by

$m_{1,2}=\left(-p\pm\sqrt{p^2-4q}\right)/2,$ (14)

which, once more, are not real due to the fact that b2 – 4ac < 0 implies

$$p^2-4q \prec 0.$$

Now, motivated by the form of the roots of the auxiliary equation (14), $\text{set } \varphi(x)=y(x)\cdot e^{-\frac{p}{2}x}$ and differentiate twice to obtain

$$\varphi'=y'e^{-\frac{p}{2}x}-\frac{p}{2}ye^{-\frac{p}{2}x}$$

And

$$\varphi''=y''e^{-\frac{p}{2}x}-py'e^{-\frac{p}{2}x}+\frac{p^2}{4}ye^{-\frac{p}{2}x}.$$

By substituting the above into equation (12) we derive

$y''+\left(q-\frac{p^2}{4}\right)y=0,$ (15)

which is an ODE of the form (7), with $\omega^2=(4q-p^2)/4.$

Moreover, $\omega = \sqrt{(4q - p^2)}/2$, with the root $\sqrt{4q - p^2}$ being real, since $4q - p^2 > 0$. Therefore, the general solution of equation (12) is given by

$$\varphi(x) = e^{-\frac{p}{2}x}[\alpha \cos \omega x + \beta \sin \omega x], \tag{16}$$

where $\omega = \sqrt{(4q - p^2)}/2.$

We illustrate the above approach by the following simple example.

EXAMPLE 1 Consider the second-order differential equation

$$\varphi'' + 6\varphi' + 13\varphi = 0.$$

Here, p = 6, q = 13, p2 – 4q = –16 and ! = 2. Hence, the general solution to the above differential equation, according to (16), is

$$\varphi(x) = e^{-3x}(\alpha \cos 2x + \beta \sin 2x).$$

If p and q are functionalities of x for the homogenous equation ( 12), various approaches should be taken into consideration. A system was recently developed based on transformations and repeated iterated integration (Wilmer & Costa, 2007). This method can also be extended to an ODE without relying on complex numbers with constant coefficients. But it extracts only one of the two lines of freedom and not the entire solution.

**4. Inhomogeneous Second-Order Odes With Constant Coefficients**

In this section, we examine the inhomogeneous second-order linear ODE

$$\varphi''(x) + p\varphi'(x) + q\varphi(x) = f(x), \tag{17}$$

Where f(x) functions continuously, p and q consist in p2 – 4q < 0.

This solution is called the equation complementary function (17), and is shown by 'c(x). This is the case f(x) = 0.

If 'p(x) is a solution of the inhomogeneous equation (17), called a particular solution, then the overall solution of (17) is given

$$\varphi(x) = \varphi_c(x) + \varphi_p(x). \tag{18}$$

We will illustrate how to find a particular solution using a general method known as variation of parameters as follows.

Suppose y1 and y2 are two linearly independent solutions of equation (12) and try to find a particular solution of (17) of the form

$$\varphi_p = r \cdot y_1 + z \cdot y_2, \tag{19}$$

where r and z are differentiable functions in the variable x.

By differentiating relation (19) twice, we get

$$\varphi_p' = r'y_1 + ry_1' + z'y_2 + zy_2'$$

And

$$\varphi_p'' = (r'y_1' + z'y_2') + (ry_1'' + zy_2'') + (r'y_1 + z'y_2)'.$$

We now substitute the above into (17) and after rearranging terms, we derive

$$r(y_1'' + py_1' + qy_1) + z(y_2'' + py_2' + qy_2) + p(r'y_1 + z'y_2) + (r'y_1' + z'y_2') + (r'y_1 + z'y_2)' = f(x). \tag{20}$$

But $y_1'' + py_1' + qy_1 = y_2'' + py_2' + qy_2 = 0$, since y1 and y2 are solutions of equation (12) and thus, from (20) we get

$$p(r'y_1 + z'y_2) + (r'y_1' + z'y_2') + (r'y_1 + z'y_2)' = f(x). \tag{21}$$

In order for equation (21) to hold, the functions r and z can be chosen to satisfy the system of equations

$$\begin{cases} r'y_1 + z'y_2 = 0 \\ r'y_1' + z'y_2' = f \end{cases}. \tag{22}$$

This system can always be solved uniquely for r0 and z0 , since y1 and y2 are linearly independent and so their Wronskian does not vanish, i.e.

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0.$$

Finally, r and z can be determined by integration.

We illustrate the above approach by a simple example.

EXAMPLE 2

Find the general solution of the inhomogeneous ODE

$$\varphi''(x) + \varphi(x) = \csc x. \tag{23}$$

Here p = 0 and q =1, so the complementary function of (23) is

$$\varphi_c(x) = c_0 \cos x + c_1 \sin x$$

Also y1 = cos x and y2 = sin x, thus the system (22) in this case is

$$\begin{cases} r' \cos x + z' \sin x = 0 \\ -r' \sin x + z' \cos x = \csc x \end{cases}.$$

Solving for r0 and z0 , we get

$$r' = -1 \text{ and } z' = \cot x,$$

from which

$$r = -\int dx = -x \text{ and } z = \int \cot x dx = \ln|\sin x|.$$

Hence, the general solution of equation (23) is given by

$$\varphi(x) = c_0 \cos x + c_1 \sin x - x \cos x + \sin x \ln|\sin x|.$$

## 5. Conclusion

The subject of differential equations falls from the last year of high school, up to tertiary education, in several mathematical, physical and engineering courses. This subject requires a strong mathematical history for the participating students and a resourceful instructor. Secondary homogenous and non-homogenous ODEs with constant coefficients play an important role in the explanation of certain physical anomalies commonly seen in secondary and first-year groups. Although students enrolled in these courses may have a clear understanding of basic computation, they may not have had a complex variable analysis. The goal of this note was to present an alternative method to obtain general solutions without using complex numbers for these differential equations.

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