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A Walk Through (3, 2)-Jection Operator

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ABSTRACT

In this present paper, we focus our attention to establish some relationship between (3,2)-jection operators and Trijection with some positive integral powers.

INTRODUCTION

Analogous to the concepts of projection, Trijection, Pentajection, we introduce the concept of(3, 2)-jection operator, which is a suitable generalization of projection. We now turn our attention to (3, 2)-jection operator with positive integral power of index and we give a complete discussion of the relationship among projection, Trijection, and (3, 2)-jection operator.

IMPORTANT DEFINITIONS :

(1) **Linear operator:** It is an operator E on a linear space L such that E(ax + by) = a E(x) + bE(y) $\forall x, y \in L$ and for scalars a and b.

(2) **Projection operator:** It is an operator E on some subspace M of a linear space L such that $E^2 = E$.

(3) **Trijection operator:** It is the linear operator E on a linear space L such that $E^3 = E$.

(4) **(3, 2)-jection operator:** It is the linear operator E on a linear space L such that$E^3 = E^2$.

MAIN RESULTS :

Theorem 1 :

Let p and q be any two scalars such that p + q = 1 and E be a (3, 2)-jection then

$$(pE+qE^2)^n = E^2 \text{ for } 1 < n \in N$$

Proof : We denote the statement $(pE+qE^2)^n = E^2$ by P(n).

For n = 2, we have

$$\begin{aligned}(pE+qE^2)^2 &= p^2E^2 + q^2E^4 + 2pqE^3 \\ &= p^2E^2 + q^2E^2 + 2pqE^2 \qquad \{\because E^3 = E^2 \,\&\, E^4 = E^2\} \\ &= (p^2+q^2+2pq)E^2 \\ &= (p+q)^2E^2 \\ &= E^2 \qquad (\text{putting } p+q=1)\end{aligned}$$

Hence P(2) is true

Now, we suppose that P(k) be true $\forall\ 2 \le k < N$

$$\text{i.e.} \quad (pE+qE^2)^k = E^2 \qquad \ldots\ldots (1.1)$$

For n = k + 1, we have

$$\begin{aligned}(pE+qE^2)^{k+1} &= (pE+qE^2)^k(pE+qE^2) \\ &= E^2(pE+qE^2) \qquad \{\text{from } (1.1)\} \\ &= pE^3 + qE^4 \\ &= pE^2 + qE^2 \qquad \{\text{since } E^3 = E^2, E^4 = E^2\} \\ &= (p+q)E^2 \\ &= E^2 \qquad (\text{putting } p+q=1)\end{aligned}$$

Hence P(k) is true $\Rightarrow$ P(k + 1) is true. Thus by induction P(n) is true $\forall\, 2 \le n \in N$

Theorem 2:

If E be a trijection then

$$\left\{\frac{1}{2}(E+E^2)\right\}^n = \frac{1}{2}(E+E^2)\ \forall\ n \in N.$$

Proof : Since E is a trijection then from the definition, we have

$$E^3 = E \quad \text{..... (2.1)}$$

Now,

$$E^4 = E^3 \cdot E$$
$$= E \cdot E \quad \{\text{from (2.1)}\}$$
$$= E^2 \quad \text{..... (2.2)}$$

Here, we prove the theorem by the method of induction, so for convenience we denote the statement "$\left\{\frac{1}{2}(E+E^2)\right\}^n = \frac{1}{2}(E+E^2)$" by P(n).

For n = 1, we have

$\left\{\frac{1}{2}(E+E^2)\right\}^1 = \frac{1}{2}(E+E^2)$ which is clearly true.

$\Rightarrow$ P(1) is true.

For n =2 , we have

$$\left\{\frac{1}{2}(E+E^2)\right\}^2 = \frac{1}{4}(E+E^2)^2$$
$$= \frac{1}{4}(E^2+E^4+2E^3)$$
$$= \frac{1}{4}(E^2+E^2+2E)$$
$$= \frac{1}{2}(E^2+E) \quad \text{... (2.3)}$$

$\Rightarrow$ P(2) is true.

Now, we suppose P(k) be true $\forall\ 1 \le k < n \in N$ then

$$\left\{\frac{1}{2}(E+E^2)\right\}^k = \frac{1}{2}(E^2+E) \quad \text{... (2.4)}$$

For n = k + 1, we have

$$\left\{\frac{1}{2}(E+E^2)\right\}^{k+1} = \left\{\frac{1}{2}(E^2+E)\right\}^k \left\{\frac{1}{2}(E+E^2)\right\}$$
$$= \frac{1}{2}(E^2+E)\frac{1}{2}(E+E^2) \quad \{\text{from (2.4)}\}$$
$$= \left\{\frac{1}{2}(E+E^2)\right\}^2$$
$$= \frac{1}{2}(E+E^2) \quad \{\text{from (2.3)}\}$$

Hence, P(k) is true $\Rightarrow$ P(k+1) is true.

So by the use of principal of induction P(n) is true $\forall\ n \in N$

Thus, $\left\{\frac{1}{2}(E+E^2)\right\}^n = \frac{1}{2}(E+E^2)$

In particular,

For n = 2, $\frac{1}{2}(E+E^2)$ is a projection

For n = 3, $\frac{1}{2}(E+E^2)$ is a trijection.

Also, $\left[\frac{1}{2}(E+E^2)^2\right]^3 = \frac{1}{2}(E+E^2)$ for n = 3

And $\left[\frac{1}{2}(E+E^2)^2\right]^2 = \frac{1}{2}(E+E^2)$ for n = 2

$\Rightarrow \left[\frac{1}{2}(E+E^2)^2\right]^3 = \left[\frac{1}{2}(E+E^2)^2\right]^2$

$\Rightarrow \frac{1}{2}(E+E^2)$ is a (3, 2) jection.

Theorem 3 :

If E be a trijection then $\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)\ \forall n \in N$.

Proof :
Since E is a trijection then from the definition,
we have

$$E^3 = E \quad \ldots (3.1)$$

and
$$E^4 = E^3.E = EE \ \{\text{from } (3.1)\} = E^2 \quad \ldots (3.2)$$

Now, we prove the theorem by the method of induction.

Here for convenience, we denote the statement "$\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)\ \forall n \in N$," by P(n).

For n = 1, we have

$$\frac{1}{2}(E^2-E) = \frac{1}{2}(E^2-E) \text{ which is trivial.}$$

Hence P(1) is true.
For n = 2, we have

$$\left[\frac{1}{2}(E^2-E)\right]^2 = \frac{1}{4}(E^4+E^2-2E^3)$$

$$= \frac{1}{4}(E^2+E^2-2E) \quad \{\text{from (3.1) and (3.2)}\}$$

$$= \frac{1}{4}(2E^2-2E)$$

$$= \frac{1}{2}(E^2-E) \quad \ldots (3.3)$$

$\Rightarrow$ P(2) is true.

Now, we suppose that P(k) be true $\forall 1 \le k < n \in N$ then

$$\left[\frac{1}{2}(E^2-E)\right]^k = \frac{1}{2}(E^2-E) \quad \ldots (3.4)$$

For n = k + 1, we have

$$\left[\frac{1}{2}(E^2-E)\right]^{k+1} = \left[\frac{1}{2}(E^2-E)\right]^k \left[\frac{1}{2}(E^2-E)\right]$$

$$= \left[\frac{1}{2}(E^2-E)\right]\left[\frac{1}{2}(E^2-E)\right] \quad \{\text{from (3.4)}\}$$

$$= \frac{1}{2}(E^2-E) \quad \{\text{from (3.3)}\}$$

So, P(k) is true $\Rightarrow$ P(k + 1) is true
Hence by the use of method of induction P(n) is true $\forall n \in N$.

i.e. $\left[\frac{1}{2}(E^2-E)\right]^n = \frac{1}{2}(E^2-E)$

In particular,

For n = 2, $\frac{1}{2}(E^2-E)$ is a projection

For n = 3, $\frac{1}{2}(E^2-E)$ is a trijection

Also, $\left[\frac{1}{2}(E^2 - E)\right]^3 = \left[\frac{1}{2}(E^2 - E)\right]^2$

So, $\frac{1}{2}(E^2 - E)$ is a (3, 2)-jection.

Theorem 4. :

If E be a trijction then $(I - E)^n = I - E^2 \ \forall n \in N$.

Proof :

Since E is a trijection then from the definition, we have

$$E^3 = E \quad \ldots (4.1)$$

and

$$\begin{aligned} E^4 &= E^3.E \\ &= EE \quad \{\text{from } (4.1)\} \\ &= E^2 \quad \ldots (4.2) \end{aligned}$$

Now, we prove the theorem by the method of induction.

So, for convenience, we denote the statement

"$(I - E^2)^n = I - E^2 \ \forall n \in N$, by P(n).

For n = 1, we have

$I - E^2 = I - E^2$ which is trivial

$\Rightarrow$ P(1) is true

For n = 2, we have

$$\begin{aligned} (I - E^2)^2 &= I + E^4 - 2E^2 \\ &= I + E^2 - 2E^2 \quad \{\text{from } (4.2)\} \\ &= I - E^2 \quad \ldots (4.3) \end{aligned}$$

$\Rightarrow$ P(2) is true

Now, we suppose P(k) be true $1 \le k < n \in N$ then

$$(I - E^2)^k = I - E^2 \quad \ldots (4.4)$$

For n = k + 1, we have

$$\begin{aligned} (I - E^2)^{k+1} &= (I - E^2)^k(I - E^2) \\ &= (I - E^2)(I - E^2) \quad \{\text{from } (4.4)\} \\ &= (I - E^2)^2 \\ &= I - E^2 \quad \{\text{from } (4.3)\} \end{aligned}$$

Thus, P(k) is true

$\Rightarrow$ P(k + 1) is true

So by applying the principle of induction

P(n) is true $\forall n \in N$

This means,

$(I - E^2)^n = I - E^2 \ \forall n \in N$

In particular,

For n = 2, $I - E^2$ is a projection

For n = 3, $I - E^2$ is a trijection

Also, $(I - E^2)^3 = (I - E^2)^2$

$\Rightarrow$ $I - E^2$ is a (3, 2)-jection

Theorem 5 :

If E be a trijection then

$$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I + \frac{1}{2}E - \frac{1}{2}E^2 \ \forall n \in N$$

Proof :

Since E is a trijection, then from the definition, we have

$$E^3 = E \quad \ldots (5.1)$$

and

$$\begin{aligned} E^4 &= E^3.E \\ &= EE \quad \{\text{from } (5.1)\} \\ &= E^2 \quad \ldots (5.2) \end{aligned}$$

Now, we prove the theorem by the method of induction.

So for convenience, we denote the statement

"$\left(I + \frac{1}{2}E - \frac{1}{2}E^2\right)^n = I + \frac{1}{2}E - \frac{1}{2}E^2$" by P(n).

For n = 1, we have

$I + \frac{1}{2}E - \frac{1}{2}E^2 = I + \frac{1}{2}E - \frac{1}{2}E^2$ which is trivial.

$\Rightarrow$ P(1) is true

For n = 2, we have

$$\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^2 = I+\frac{1}{4}E^2+\frac{1}{4}E^4+E-\frac{1}{2}E^3-E^2$$

$$= I+\frac{1}{4}E^2+\frac{1}{4}E^2+E-\frac{1}{2}E-E^2 \qquad \{\text{from (5.1) and (5.2)}\}$$

$$= I+\frac{1}{2}E-\frac{1}{2}E^2 \qquad \ldots(5.3)$$

From (5.3), P(2) is true

Now, we suppose that P(k) be true for $k \in N$ such that $1 \leq k \leq n \in N$ then

$$\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^k = I+\frac{1}{2}E-\frac{1}{2}E^2 \qquad \ldots (5.4)$$

For n = k + 1,
we have

$$\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^{k+1} = \left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^k\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)$$

$$= \left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right) \qquad \{\text{from (5.4)}\}$$

$$= \left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^2$$

$$= I+\frac{1}{2}E-\frac{1}{2}E^2 \qquad \{\text{from 5.3}\}$$

Thus, P(k) is true $\Rightarrow$ P(k + 1) is true

i.e. P(n) is true $\forall n \in N$ by the use of induction

Hence,

$$\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^n = \left(I+\frac{1}{2}E-\frac{1}{2}E^2\right) \ \forall n \in N$$

In particular,

For n = 2, $I+\frac{1}{2}E-\frac{1}{2}E^2$ is a projection

For n = 3, $I+\frac{1}{2}E-\frac{1}{2}E^2$ is trijection

Also,

$$\left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^3 = \left(I+\frac{1}{2}E-\frac{1}{2}E^2\right)^2$$

$\Rightarrow$ $I+\frac{1}{2}E-\frac{1}{2}E^2$ is a (3, 2)-jection

Theorem 6 :

If E be a trijection then

$$\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^n = I-\frac{1}{2}E-\frac{1}{2}E^2 \ \forall n \in N$$

Proof : Since E is a trijection then

$$E^3 = E \qquad \ldots (6.1)$$

and

$$E^4 = E^3.E$$

$$= E \cdot E \qquad \{\text{from (6.1)}\}$$

$$= E^2 \qquad \ldots (6.2)$$

Here, we prove the theorem by the use of method of induction.

Now for the convenience, we denote the statement "$\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^n = I-\frac{1}{2}E-\frac{1}{2}E^2 \ \forall n \in N$" by P(n)

It is obvious that P(1) is true

For n = 2, we have

$$\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^2 = I+\frac{1}{4}E^2+\frac{1}{4}E^4-E+\frac{1}{2}E^3-E^2$$

$$= I+\frac{1}{4}E^2+\frac{1}{4}E^2-E+\frac{1}{2}E-E^2 \qquad \{\text{from (6.1) \& (6.2)}\}$$

$$= I-\frac{1}{2}E-\frac{1}{2}E^2 \qquad \ldots (6.3)$$

From (6.3), it is clear that P(2) is true

Now, we suppose that P(k) be true for $k \in N$ such that $1 \le k < n \,\forall n \in N$

then

$$\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^k = I-\frac{1}{2}E-\frac{1}{2}E^2 \qquad \ldots (6.4)$$

For n = k + 1, we have

$$\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^{k+1} = \left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^k\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)$$

$$= \left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)$$

{from (6.4)}

$$= \left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^2$$

$$= I-\frac{1}{2}E-\frac{1}{2}E^2 \quad \{\text{from (6.3)}$$

Hence, P(k) is true $\Rightarrow$ P(k + 1) is true

So, by the use of methodof induction that P(n) is true $\forall n \in N$

Thus, $\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^n = I-\frac{1}{2}E-\frac{1}{2}E^2$

In particular,

For n = 2, $I-\frac{1}{2}E-\frac{1}{2}E^2$ is a projection

For n = 3, $I-\frac{1}{2}E-\frac{1}{2}E^2$ is a trijcetion

Also, $\left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^3 = \left(I-\frac{1}{2}E-\frac{1}{2}E^2\right)^2$

$\Rightarrow$ $I-\frac{1}{2}E-\frac{1}{2}E^2$ is a (3,2)-jection.

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