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A Study of Linear Space - Geometryof Banach Spaces

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ABSTRACT

In mathematics, space is an unbounded continuum (unbroken set of points) in which exactly three numerical coordinates are necessary to uniquely define the location of any particular point. It is sometimes called 3-D space because it contains three distance dimensions. If a continuum requires fewer or more than three coordinates (dimensions) to uniquely define the location of a point, that continuum is sometimes called n-space or n-dimensional space, where n is the number of dimensions. Thus, for example, a line constitutes 1-space and a plane constitutes 2-space. When time is considered as a dimension along with the usual three in conventional space, the result is sometimes called 4-space, 4-dimensional space, time-space, or space-time. Spatial understandings are necessary for interpreting, understanding, and appreciating our inherently geometric world. Insights and intuitions about two- and three-dimensional shapes and their characteristics, the interrelationships of shapes, and the effects of changes to shapes are important aspects of spatial sense. Children who develop a strong sense of spatial relationships and who master the concepts and language of geometry are better prepared to learn number and measurement idea. Arithmetic is an important corner of mathematics, but too often we neglect the rest of the field. Geometry suffers because we have the mistaken impression that it doesn't become real, serious mathematics until it gets abstract and we deal with proof. But geometry is important, even in its less formal form.

INTRODUCTION

A **linear space** is a basic structure in incidence geometry. A linear space consists of a set of elements called **points**, and a set of elements called **lines**. Each line is a distinct subsetof the points. The points in a line are said to be **incident** with the line. Any two lines may have no more than one point in common. Intuitively, this rule can be visualized as two straight lines, which never intersect more than once.

(Finite) linear spaces can be seen as a generalization of projective and affine planes, and more broadly, of block designs, where the requirement that every block contains the same number of points is dropped and the essential structural characteristic is that 2 points are incident with exactly 1 line.

The term *linear space* was coined by Libois in 1964, though many results about linear spaces are much older.

Definition

Let $L = (P, G, I)$ be an incidence structure, for which the elements of P are called points and the elements of G are called lines. L is a *linear space* if the following three axioms hold:

- (L1) two points are incident with exactly one line.
- (L2) every line is incident to at least two points.
- (L3) L contains at least two lines.

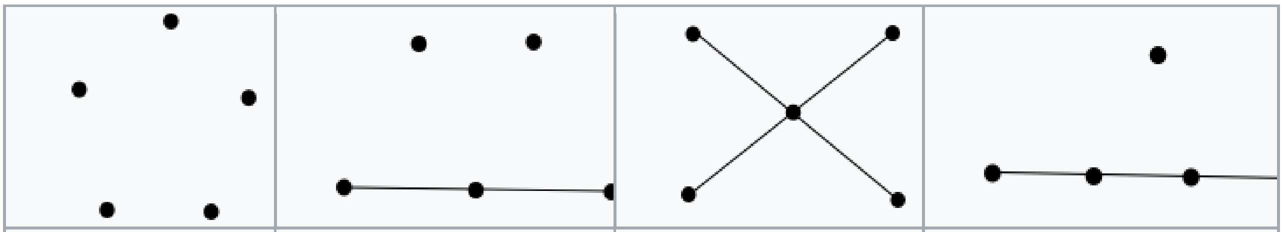
Some authors drop (L3) when defining linear spaces. In such a situation the linear spaces complying to (L3) are considered as *nontrivial* and those who don't as *trivial*.

Examples

The regular Euclidean plane with its points and lines constitutes a linear space, moreover all affine and projective spaces are linear spaces as well.

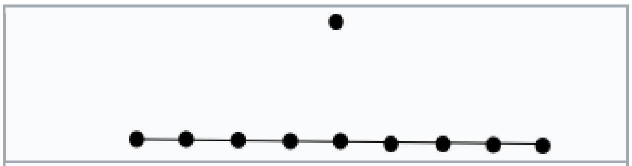
The table below shows all possible nontrivial linear spaces of five points. Because any two points are always incident with one line, the lines being incident with only two points are not drawn, by convention. The trivial case is simply a line through five points.

In the first illustration, the ten lines connecting the ten pairs of points are not drawn. In the second illustration, seven lines connecting seven pairs of points are not drawn.



10 lines	8 lines	6 lines	5 lines

A linear space of n points containing a line being incident with $n - 1$ points is called a *near pencil*. (See pencil)



near pencil with 10 points

BLOCK DESIGN

In combinatorial mathematics, a **block design** is a set together with a family of subsets (repeated subsets are allowed at times) whose members are chosen to satisfy some set of properties that are deemed useful for a particular application. These applications come from many areas, including experimental design, finite geometry, physical chemistry, software testing, cryptography, and algebraic geometry. Many variations have been examined, but the most intensely studied are the **balanced incomplete block designs** (BIBDs or 2-designs) which historically were related to statistical issues in the design of experiments.[1][2]

A block design in which all the blocks have the same size is called *uniform*. The designs discussed in this article are all uniform. Pairwise balanced designs (PBDs) are examples of block designs that are not necessarily uniform.

Definition of a BIBD (or 2-design)

Given a finite set X (of elements called *points*) and integers $k, r, \lambda \geq 1$, we define a *2-design* (or *BIBD*, standing for balanced incomplete block design) B to be a family of k-element subsets of X, called *blocks*, such that any x in X is contained in r blocks, and any pair of distinct points x and y in X is contained in λ blocks.

"Family" in the above definition can be replaced by "set" if repeated blocks are not allowed. Designs in which repeated blocks are not allowed are called *simple*.

Here v (the number of elements of X, called points), b (the number of blocks), k, r, and λ are the *parameters* of the design. (To avoid degenerate examples, it is also assumed that $v > k$, so that no block contains all the elements of the set. This is the meaning of "incomplete" in the name of these designs.) In a table:

v	points, number of elements of X
b	number of blocks
r	number of blocks containing a given point
k	number of points in a block
λ	number of blocks containing any 2 (or more generally t) distinct points

The design is called a (v, k, λ)-design or a (v, b, r, k, λ)-design. The parameters are not all independent; v, k, and λ determine b and r, and not all combinations of v, k, and λ are possible. The two basic equations connecting these parameters are obtained by counting the number of pairs (B, p) where B is a block and p is a point in that block, and obtained from the count of triples (p, q, B) where p and q are distinct points and B is a block that contains them both, and dividing this count by v.

These conditions are not sufficient as, for example, a (43,7,1)-design does not exist.[3]

The *order* of a 2-design is defined to be $n = r - \lambda$. The **complement** of a 2-design is obtained by replacing each block with its complement in the point set X. It is also a 2-design and has parameters $v' = v$, $b' = b$, $r' = b - r$, $k' = v - k$, $\lambda' = \lambda + b - 2r$. A 2-design and its complement have the same order.

A fundamental theorem, Fisher's inequality, named after the statistician Ronald Fisher, is that $b \geq v$ in any 2-design.

Examples

The unique (6,3,2)-design has 10 blocks (b = 10) and each element is repeated 5 times (r = 5).[4] Using the symbols 0 − 5, the blocks are the following triples:

012 013 024 035 045 125 134 145 234 2 35.

One of four nonisomorphic (8,4,3)-designs has 14 blocks with each element repeated 7 times. Using the symbols 0 − 7 the blocks are the following 4-tuples:[4]

0123 0124 0156 0257 0345 0367 0467 1267 1346 1357 1457 2347 2356 2456.

The unique (7,3,1)-design has 7 blocks with each element repeated 3 times. Using the symbols 0 − 6, the blocks are the following triples:[4]

013 026 045 124 156 235 346. If the elements are thought as points on a Fano plane, then these blocks are the lines.

Symmetric BIBDs

The case of equality in Fisher's inequality, that is, a 2-design with an equal number of points and blocks, is called a **symmetric design**.[5] Symmetric designs have the smallest number of blocks among all the 2-designs with the same number of points.

In a symmetric design $r = k$ holds as well as $b = v$, and, while it is generally not true in arbitrary 2-designs, in a symmetric design every two distinct blocks meet in λ points.[6] A theorem of Ryser provides the converse. If X is a v-element set, and B is a v-element set of k-element subsets (the "blocks"), such that any two distinct blocks have exactly λ points in common, then (X, B) is a symmetric block design.[7]

The parameters of a symmetric design satisfy

This imposes strong restrictions on v, so the number of points is far from arbitrary. The Bruck–Ryser–Chowla theorem gives necessary, but not sufficient, conditions for the existence of a symmetric design in terms of these parameters.

The following are important examples of symmetric 2-designs:

Projective planes

Finite projective planes are symmetric 2-designs with λ = 1 and order $n > 1$. For these designs the symmetric design equation becomes:

Since $k = r$ we can write the *order of a projective plane* as $n = k - 1$ and, from the displayed equation above, we obtain $v = (n + 1)n + 1 = n^2 + n + 1$ points in a projective plane of order n.

As a projective plane is a symmetric design, we have $b = v$, meaning that $b = n^2 + n + 1$ also. The number b is the number of *lines* of the projective plane. There can be no repeated lines since λ = 1, so a projective plane is a simple 2-design in which the number of lines and the number of points are always the same. For a projective plane, k is the number of points on each line and it is equal to $n + 1$. Similarly, $r = n + 1$ is the number of lines with which a given point is incident.

For $n = 2$ we get a projective plane of order 2, also called the Fano plane, with $v = 4 + 2 + 1 = 7$ points and 7 lines. In the Fano plane, each line has $n + 1 = 3$ points and each point belongs to $n + 1 = 3$ lines.

Projective planes are known to exist for all orders which are prime numbers or powers of primes. They form the only known infinite family (with respect to having a constant λ value) of symmetric block designs.[8]

Biplanes

A **biplane** or **biplane geometry** is a symmetric 2-design with $\lambda = 2$; that is, every set of two points is contained in two blocks ("lines"), while any two lines intersect in two points.[8]They are similar to finite projective planes, except that rather than two points determining one line (and two lines determining one point), two points determine two lines (respectively, points). A biplane of order n is one whose blocks have $k = n + 2$ points; it has $v = 1 + (n + 2)(n + 1)/2$ points (since $r = k$).

The 18 known examples[9] are listed below.

- (Trivial) The order 0 biplane has 2 points (and lines of size 2; a 2-(2,2,2) design); it is two points, with two blocks, each consisting of both points. Geometrically, it is the digon.
- The order 1 biplane has 4 points (and lines of size 3; a 2-(4,3,2) design); it is the complete design with $v = 4$ and $k = 3$. Geometrically, the points are the vertices and the blocks are the faces of a tetrahedron.
- The order 2 biplane is the complement of the Fano plane: it has 7 points (and lines of size 4; a 2-(7,4,2)), where the lines are given as the *complements* of the (3-point) lines in the Fano plane.[10]
- The order 3 biplane has 11 points (and lines of size 5; a 2-(11,5,2)), and is also known as the **Paley biplane** after Raymond Paley; it is associated to the Paley digraph of order 11, which is constructed using the field with 11 elements, and is the Hadamard 2-design associated to the size 12 Hadamard matrix; see Paley construction I.

Algebraically this corresponds to the exceptional embedding of the projective special linear group $PSL(2,5)$ in $PSL(2,11)$ – see projective linear group: action on p points for details.[11]

- There are three biplanes of order 4 (and 16 points, lines of size 6; a 2-(16,6,2)). One is the Kummer configuration. These three designs are also Menon designs.
- There are four biplanes of order 7 (and 37 points, lines of size 9; a 2-(37,9,2)).[12]
- There are five biplanes of order 9 (and 56 points, lines of size 11; a 2-(56,11,2)).[13]
- Two biplanes are known of order 11 (and 79 points, lines of size 13; a 2-(79,13,2)).[14]

Biplanes of orders 5, 6, 8 and 10 do not exist, as shown by the Bruck-Ryser-Chowla theorem.

Hadamard 2-designs

An Hadamard matrix of size m is an $m \times m$ matrix **H** whose entries are ±1 such that $\mathbf{HH}^{\top} = m\mathbf{I}_m$, where $\mathbf{H}^{\top}$ is the transpose of **H** and $\mathbf{I}_m$ is the $m \times m$ identity matrix. An Hadamard matrix can be put into *standardized form* (that is, converted to an equivalent Hadamard matrix) where the first row and first column entries are all +1. If the size $m > 2$ then m must be a multiple of 4.

Given an Hadamard matrix of size $4a$ in standardized form, remove the first row and first column and convert every −1 to a 0. The resulting 0–1 matrix **M** is the incidence matrix of a symmetric 2-$(4a - 1, 2a - 1, a - 1)$ design called an **Hadamard 2-design**.[15] It contains blocks/points; each contains/is contained in points/blocks. Each pair of points is contained in exactly blocks.

This construction is reversible, and the incidence matrix of a symmetric 2-design with these parameters can be used to form an Hadamard matrix of size 4*a*.

Resolvable 2-designs

A **resolvable 2-design** is a BIBD whose blocks can be partitioned into sets (called *parallel classes*), each of which forms a partition of the point set of the BIBD. The set of parallel classes is called a *resolution* of the design.

If a 2-(v,k,λ) resolvable design has c parallel classes, then $b \geq v + c - 1$.[16]

Consequently, a symmetric design can not have a non-trivial (more than one parallel class) resolution.[17]

Archetypical resolvable 2-designs are the finite affine planes. A solution of the famous 15 schoolgirl problem is a resolution of a 2-(15,3,1) design.[18]

Generalization: t-designs

Given any positive integer t, a t-design B is a class of k-element subsets of X, called *blocks*, such that every point x in X appears in exactly r blocks, and every t-element subset T appears in exactly λ blocks. The numbers v (the number of elements of X), b (the number of blocks), k, r, λ, and t are the *parameters* of the design. The design may be called a t-(v,k,λ)-design. Again, these four numbers determine b and r and the four numbers themselves cannot be chosen arbitrarily. The equations are

where λ_i is the number of blocks that contain any i-element set of points.

Theorem:[19] Any t-(v,k,λ)-design is also an s-(v,k,λ_s)-design for any s with $1 \leq s \leq t$. (Note that the "lambda value" changes as above and depends on s.)

A consequence of this theorem is that every t-design with $t \geq 2$ is also a 2-design.

A t-(v,k,1)-design is called a Steiner system.

The term *block design* by itself usually means a 2-design.

Conclusion

Geometry, the branch of mathematics concerned with the shape of individual objects, spatial relationships among various objects, and the properties of surrounding space. It is one of the oldest branches of mathematics, having arisen in response to such practical problems as those found in surveying, and its name is derived from Greek words meaning “Earth measurement.”Eventually it was realized that geometry need not be limited to the study of flat surfaces (plane geometry) and rigid three-dimensional objects (solid geometry) but that even the most abstract thoughts and images might be represented and developed in geometric terms. Modern mathematics uses many types of spaces, such as Euclidean spaces, linear spaces, topological spaces, Hilbert spaces, or probability spaces, it does not define the notion of "space" itself.

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