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Inequalities for the Polar Derivative of a Polynomial with Confined Zeros

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ABSTRACT

In this paper certain irregular characteristics for the polar derivative of a polynomial with bound zeros are given, which summarize and re one some notable polynomial inequalities due to Govil, Malik, Aziz and others. In this paper we reach out above imbalance for the polar subsidiary of polynomials having no zeros in |z| < 1, aside from s-overlay zeros at the birthplace. . On the off chance that the quantity of zeros isn't known at the beginning, the outcome might be untrustworthy as in a portion of the zeros may not be found. Then again, if the locale contains no zeros one may scan long and unproductively for something which doesn't exist. Regardless of whether the quantity of zeros in the district is known, it might be hard to urge the emphasess into merging, or to stop them continually combining to a similar zero. Our outcome sums up certain notable polynomial inequalities.

Introduction

The idea of best estimation was introduced into mathematical analysis by crafted by the renowned Russian mathematician P.L. Chebyshev (1821-1894), who concentrated a portion of the properties of polynomials with deviation from a given consistent capacity. As per Telyakovskiy, "Among those that are fundamental in Approximation Theory are the Extremal issues associated with inequalities for the polar subordinate of polynomials. The utilization of inequalities of this sort is a fundamental method in evidences of converse issues of approximation hypothesis "In a prize winning exposition as an issue of best estimation, Bernstein demonstrated and utilized an imbalance concerning the subordinate of polynomials. He himself had the option to demonstrate that on the off chance that p(x) is a quadratic polynomial and $|p(x)| \leq$ on [-1,1], then $|p(x)| \leq 4$ on the same interval. In this paper one studies the following problem:

I. INTRODUCTION AND STATEMENT OF RESULTS

Let *p*(*z*) be a polynomial of degree n and a be any complex or real number, at that point polar derivative of p(z) as for point a, which is denoted by D_α,p(z), is defined as

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z) \qquad (1.1.1)$$

Polynomial D_α,p(z) is of degree at least *n*-1 and it sums up the normal subsidiary *p*'(*z*) of *p*(*z*) as in

$$\lim_{\alpha \to \infty} \frac{D_\alpha p(z)}{\alpha - z} = p'(z). \qquad (1.1.2)$$

Relating to the given most extreme limit polynomial p(z), we build a grouping of polar derivatives

$$D_{\alpha_1} p(z) = np(z) + (\alpha_1 - z)p'(z)$$

$$D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_k} p(z) = (n - k + 1) D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_{k-1}} p(z)$$

$$+ (\alpha_k - z)[D_{\alpha_1} \ldots D_{\alpha_{k-1}} p(z)]' \qquad k = 2, 3, \ldots n.$$

The points a_1, ... a_k, 1 ≤ k ≤ n might be equivalent or inconsistent. Like the kth standard subsidiary pk(z) of p(z), the kth polar subordinate D_{a1}, … ..$D_{akp(z)}$ of p(z) is a polynomial of degree at most n-k. Aziz demonstrated a few sharp outcomes concerning the most extreme modulus of the polar subordinate of a polynomial p(z) with confined zeros. Indeed, which stretches out Theorems G_1 and L_1 to the polar subordinate of p(z) and is likewise a minimized speculation of these outcomes.

THEOREM A₃. In the event that *p*(*z*) is a polynomial of degree m having no zeros in circle $|z| < 1$, at that point for $|z| \geq 1$,

$$|D_{\alpha_1} \ldots D_{\alpha_k} p(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{2}\left[|\alpha_1 \ldots \alpha_k z^{n-k}| + 1\right] \max_{|z|=1} |p(z)|$$

(1.1.3)

where $|a_i| \geq 1$ for 1 = 1,…. *k*. The outcome is most possible and equality in (1.1.3) holds for theTolvnomial *p*(*z*) z^n +1 . The following outcomes which is likewise due to Aziz extends Theorem L_1 to polar derivative of *p*(*z*);

THEOREM B₃. On the off chance that p(z) is polynomial of degree m having_no zeros in circle $|z| < k$, where $k \geq 1$, at that point for each complex or genuine number β with $|\beta| \geq 1$,

$$\max_{|z|=1} |D_\beta p(z)| \leq n\left[\frac{k+|\beta|}{1+k}\right]\max_{|z|=1}|p(z)| \quad (1.1.4)$$

The outcome is most ideal and equality in (1.1.4) holds for polynomial $p(z) = (z+k)^n$ with β ≥ 1. As refinement of Theorem A_3, Aziz. Furthermore, Shah demonstrated the accompanying outcome which additionally yields a compact speculation of Theorem J_1 & K_1:

THEOREM C_3. On the off chance that p(z) is a polynomial of degree m which doesn't disappear in circle │z│ < 1, at that point for │z│ ≥1,

$$|D_{\alpha_1} \ldots D_{\alpha_k} p(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{2}\Big\{(|\alpha_1\ldots\alpha_k||z|^{n-k}+1)\max_{|z|=1}|p(z)|$$
$$-(|\alpha_1\ldots\alpha_k||z|^{n-k}-1)\min_{|z|=1}|p(z)|\Big\} \quad (1.1.5)$$

Where $|\alpha_i| \geq 1$ for $i = 1,\ldots\ldots k$. The outcome is most ideal and equality in (1.1.5) holds for polynomial $p(z) = z^n+1$. For class of polynomials p(z) not evaporating in │z│< k, where k ≥ 1, Aziz and Shah demonstrated the accompanying refinement of Theorem which additionally stretches out Theorem B_3 to polar derivative of p(z).

THEOREM D_3. On the off chance that p(z) is a_polvnomial of degree m having no zeros in │z│< k, where k ≥ 1, at that point for each complex number a with │a│ ≥ 1,

$$\max_{|z|=1}|D_a p(z)| \leq \frac{n}{1+k}\Big\{(|\alpha|+k)\max_{|z|=1}|p(z)| - (|\alpha|-1)\min_{|z|=k}|p(z)|\Big\} \quad (1.1.6)$$

The outcome is sharp and the extremal polynomial is $p(z) = (z + k)^n$ with a ≥ 1. For a class of polynomials p(z) having each of the zeros in│z│≤ 1, Shah broadened Turan's (Theorem N_1) to the polar subsidiary of the polynomial p(z) and demonstrated:

THEOREM E_3, if all zeros of absolute limit polynomial p(z) lie in │z│l < 1, at that point for│a│≥ 1,

$$\max_{|z|=1}|D_\alpha p(z)| \geq \frac{n}{2}(|\alpha|-1)\max_{|z|=1}|p(z)| \quad (1.1.7)$$

The outcome is most ideal and fairness in (1.1.7) holds for $p(z) = (z-l)^n$ with genuine a. As a speculation of Theorem F3, Aziz and Rather demonstrated accompanying outcome which likewise gives a smaller speculation of Theorem P1 to polar subordinate of a polynomial p(z):

THEOREM F_3. On the off chance that p(z) is a polynomial of degree m having every one of its zeros in │z│≤ 1, where k ≤1, at that point for each genuine or complex number a with │a│≥ k,

$$\max_{|z|=1}|D_\alpha p(z)| \geq n\left(\frac{|\alpha|-k}{1+k}\right)\max_{|z|=1}|p(z)| \quad (1.1.8)$$

The outcome is most ideal and fairness in (1 1.8) holds for $p(z) = (z-k)^n$ with a≥ k . By applying Theorem D_3 to the polynomial $z^n p(1/z)$, Aziz and Shah demonstrated the accompanying outcome which is a smaller speculation of an aftereffect of Govil to the polar subsidiary of a polynomial:

THEOREM G_3. On the off chance that p(z) is a polynomiallegreerszeros in │z│≤ k ≤1 and if a is any complex or real number with │a│≤ 1 then for │z│=1.

$$|D_\alpha p(z)| \leq n\left(\frac{|\alpha|+k}{1+k}\right)\max_{|z|=1}|p(z)| - n\left(\frac{1-|\alpha|}{k^{n-1}(1+k)}\right)\min_{|z|=k}|p(z)| \quad (1.1.9)$$

The outcome is most ideal and extremal polynomial is $p(z) = (z+k)^n$ with a real number a ≥ 0. Aziz and Rather were able to prove following result:

THEOREM H_3. On the off chance that p(z) is polynomial of degree m having zeros in │z│≤ k where k ≥ 1 then for each genuine or complex number a with │α│≥ k,

$$\max_{|z|=1}|D_\alpha p(z)| > n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)| \quad (1.1.10)$$

In this part, we will initially stretch out Theorem 01 to the polar subsidiary of a polynomial and along these lines get a smaller speculation of this outcome.

THEOREM 3.1. In the event that p(z) is polynomial of degree m having every one of its zeros in │z│≤ 1then for │z│≤ 1

$$|D_{\alpha_1}D_{\alpha_2}\ldots D_{\alpha_k}p(z)| \leq \frac{n(n-1)\ldots(n-k+1)}{2}\Big\{(|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k}+1)\max_{|z|=1}|p(z)|$$
$$-(1-|\alpha_1\alpha_2\ldots\alpha_k||z|^{n-k})\min_{|z|=1}|p(z)|\Big\} \quad (1.1.11)$$

COROLLARY 1.1. On the off chance that p(z) is polynomial of degree m having every one of its zeros in │z│≤, at that point for each complex or real number a with │a│≤ l

$$| D_\alpha p(z) | \le \frac{n}{2}\left\{(| \alpha || z |^{n-1} +1) \max_{| z |= 1} | p(z) | -(1- | \alpha || z |^{n-1}) \min_{| z |= 1} | p(z) |\right\}$$

For $|z| \le 1$ (1.1.12)

$$p(z) = \left(\frac{z^n+1}{2}\right)$$

REMARK 1.1. If we put $a = 0$ in (1.1.12), we get for $|z| =1$

$$| np(z) - zp'(z) | \le \frac{n}{2}\left\{\max_{| z |= 1} | p(z) | - \min_{| z |= 1} | p(z) |\right\} \quad (1.1.13)$$

If , then from (1.1.13), we get

$$\max_{| z |= 1} | p(z) |= | p(e^{i\phi}) |$$

$$\max_{| z |= 1} | p'(z) | \ge | p'(e^{i\phi}) |, \quad | p'(e^{i\phi}) | \ge \frac{n}{2}\left[\max_{| z |= 1} | p(z) | + \min_{| z |= 1} | p(z) |\right]$$

(1.1.14)

from (1.1.14), we immediately get Theorem O_1.

If in Theorem 1.1, one get following generalization of Theorem N_1:

COROLLARY 3.2. On the off chnce that $p(z)$ is polyomial of degree m having all zeros in $|z|$ =then for $|z| \le 1$,

$$\min_{| z |= 1} | p(z) |= 0$$

$$| D_{\alpha_1} D_{\alpha_2} \ldots D_{\alpha_k} p(z) | \le \frac{n(n-1)\ldots(n-k+1)}{2}\left\{ \alpha_1\alpha_2 \ldots \alpha_k | | z |^{n-k} +1\right\} \max_{| z |= 1} | p(z) | \quad 1.1.15$$

Where $|\alpha_1| < 1$ for $1 \le i \le k$; $k \le n-1$. The outcome is most possible & equality holds for polynomial $p(z)$ = . Next, we demonstrate following speculation of $\frac{(z^n+1)}{2}$ Theorem F_3 by expecting lhat $p(z)$ has s-overlay zeros ($s \ge 0$) at origin. To, one prove:

THEOREM 3.2. One the off chance that $p(z)$ is polynomial of degree m having every one of its zeros in $|z| \le k \le 1$ with s-overlay zeros at origin, then for each complex number a with $|a| \ge k$, one have

$$\max_{| z |= 1} | D_\alpha p(z) | \ge \frac{(| \alpha | - k)(n+ks)}{1+k} \max_{| z |= 1} | p(z) |$$

(1.1.16)

Where $0 \le s \le n$, Dividing the different sides (1.1.16) by !al, letting $|\alpha|$ - ∞ and talking (1.1.2), one get following:

COROLLARY 3.3. On the off chance that $p(z)$ is polynomial of degree m havig every zeros in $|z| \le k \le 1$ with s-overlay zeros at origin

$$\max_{| z |= 1} | p'(z) | \ge \frac{n+ks}{1+k} \max_{| z |= 1} | p(z) | \quad (1.1.17)$$

The result is extremal polynomial is $p(z)= z^s(z+k)^{n-s}$, $0 \le s \le n$.

THEOREM 1.3 On the off chance that $p(z)$ = is polynomial of degrec m s.t. $| z_j| \le k_j \le 1$, $1 \le j \le n$ then for every real number a with $|\alpha| \ge t_0$

$$\max_{| z |= 1} | D_\alpha p(z) | \ge n\left(\frac{| \alpha | - t_0}{1+t_0}\right) \max_{| z |= 1} | p(z) | \quad (1.1.18)$$

where $t_0 = 1 -$ $\left(\frac{n}{\sum_{j=1}^{n} \frac{1}{1-k_j}}\right)$

The equality holds in (1.1.18) for $p(z)$ — $(z-k)^n$, where $a \ge k$ $\prod_{j=1}^{n}(z-z_j)$

REMARK 1.3 It can be easily show that Theorem 1.3 includeded as special inequality (1.1.7)

REMARK 1.4. Dividing the different sides of (1.1.18) by $|a|$, letting $|a|$ & noting (1.1.2), one get

$$\max_{| z |= 1} | p'(z) | \ge \frac{n}{1+t_0} \max_{| z |= 1} | p(z) |$$

(1.1.19)

By putting value of t_o in (1.1.19) and after simplification, one get,

$$\max_{| z |= 1} | p'(z) | \ge \frac{n}{2}\left[1+\frac{1}{1+\frac{2}{n}\sum_{j=1}^{n}\frac{k_j}{1-k_j}}\right] \max_{| z |= 1} | p(z) |$$

The above outcome was as of late demonstrated by Aziz and Ahmad. For the class of polynomials p(z) having every one of its zeros in |z| ≤ k, where k ≥ 1, we demonstrate the accompanying refinement of (3.1.10), which likewise stretches out Theorem T_1 to the polar subsidiary of p(z).

$$\max_{|z|=1}|D_\alpha p(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\left\{\max_{|z|=1}|p(z)| + \frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}$$
$$+\left(1-\frac{1}{k^2}\right)|na_0+\alpha a_1|, \text{ if } n>2 \quad (1.1.20)$$

$$\max_{|z|=1}|D_\alpha p(z)| \geq n\left(\frac{|\alpha|-k}{1+k^n}\right)\left\{\max_{|z|=1}|p(z)| + \frac{|a_1|}{2k}(k-1)^2\right\}$$
$$+\left(1-\frac{1}{k}\right)|na_0+\alpha a_1|, \text{ if } n=2 \quad (1.1.21)$$

REMARK 1.5. Dividing different sides of (1.1.20) and (1.1.21) by |a|, letting |a| → ∞ and nothing (1.1.2), one get Theorem T_1.

Then again Malik have acquire a generlazation of Theorem N1 is the feeling that the correct hand side of (1.1.14) is supplanted by a factor including the indispensable mean of |p(z)| on |α| =1. Truth be told, he demonstrated that on the off chance that p(z) has every one of its zeros in |z| ≤ 1, at that point for every r > 0,

$$n\left\{\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right\}^{1/r} \leq \left\{\int_0^{2\pi}|1+e^{i\theta}|^r\,d\theta\right\}^{1/r}\max_{|z|=1}|p'(z)| \quad (1.1.22)$$

If one let *r* →∞ in (1.1.22) and make utilized of well known fact from analysis.

$$\left\{\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right\}^{1/r} \to \max_{0\leq\theta<2\pi}|p(e^{i\theta})| \text{ as } r\to\infty, \quad (1.1.23)$$

We get Theorem N_1.One have had the option to demonstrate the accompanying generalization of Theorem E_3 as in on the correct hand side of (1.1.7) is supplanted by a factor including integral mean of |*p(z)*| on |z| =1. All the more accurately, one prove,

one $\max_{|z|=1}|p(z)$ **REMARK 1.6.** Isolating diverse side of (1.1.24) by |α|, letting |α| → ∞ and taking note of (1.1.2), get inequlity (1.1.22). For *r*→ ∞, disparity (1.1.24) decrease to (1.1.7).

Aziz expanded (1.1.22) to the class of polynomial having all their zero in |z| ≤ k, k ≤ 1 and in this way get a speculation of Theorem P_1. Truth be told, he demonstrated that if p(z) has every one of its zeros in |z| ≤ k, k ≤ 1, at that point for every r > 0,

$$n\left\{\int_0^{2\pi}\left|\frac{p(e^{i\theta})}{p'(e^{i\theta})}\right|^r d\theta\right\}^{1/r} \leq \left\{\int_0^{2\pi}|1+ke^{i\theta}|^r\,d\theta\right\}^{1/r} \quad (1.1.25)$$

Since $|p'(e^{i\theta})|$ ≤ for 0 ≤ θ ≤ 2π, inequality (1.1.25) can be replaced by

$$n\left\{\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right\}^{1/r} \leq \left\{\int_0^{2\pi}|1+ke^{i\theta}|^r\,d\theta\right\}^{1/r}\max_{|z|=1}|p'(z) \quad (1.1.26)$$

Next, one shall obtain $\max_{|z|=1}|p'(z)|$ generalization of Theorem F_3 in sense that maximum |*p(z)*| *on* |z| =1. In fact, one prove

THEOREM 1.6. On the off chance that *p(z)* is polynomial of degree *m* having zeros in |z| ≤ *k,* where *k* ≤ 1, for every complex number *a* with |α| ≥ *k* & for each r > 0,

$$n(|\alpha|-k)\left\{\int_0^{2\pi}|p(e^{i\theta})|^r\,d\theta\right\}^{1/r} \leq \left\{\int_0^{2\pi}|1+ke^{i\theta}|^r\,d\theta\right\}^{1/r}\max_{|z|=1}|D_\alpha p(z)| \quad (1.1.27)$$

The outcome is most ideal for adequately huge r and equlaity holds for p(z) = (z-k)n with a ≥ k.

Theorem 1.7. Let m = *p*(*z*.), at that point for each genuine or complex number β with |β| < 1, it follows by Rouche's Theorem that polynomial *F*(*z*) = *p*(*z*) + *βm* has every one of its zeros in |z| ≤ 1.

If $G(z) = z^n\overline{F(1/\bar{z})} = z^n\overline{p(1/\bar{z})} + \bar{\beta}mz^n = q(z) + \bar{\beta}mz^n$, then all the zeros of $G(z)$ lie in $|z| \geq 1$ and $|G(z)| = |F(z)|$ on $|z| = 1$. Therefore, for every complex number λ with $|\lambda| > 1$, the polynomial $F(z)$- λ $G(z)$ has all its zeros in $|z| \geq 1$. Hence by the repeated application of Laguerre's Theorem [91, p. 52], if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are complex numbers with $|\alpha_i| < 1$, $1 \leq i \leq k$, the polynomial $D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}(F(z)-\lambda G(z))$ has all its zeros in $|z| \geq 1$.

Equivalently, all the zeros of $D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}F(z)$ - λ $D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}G(z)$ lie in $|z| \geq 1$. This implies that for $|z| \leq 1$,

$$|D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}F(z)| \leq |D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}(G(z)|$$
$$|D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}(p(z)+\beta m)| \leq |D_{\alpha 1}D_{\alpha 2}\ldots D_{\alpha k}(q(z)+\bar{\beta}mz^n)|.$$

Picking contention of β reasonably on left hand side above and letting │β│-1 one get, the above inequality in related to lemma 1.1 gives for │z│≤, │α│<1,

$$-\left(1-|\alpha_1\alpha_2 \ . \ . \ . \ \alpha_k||z|^{n-k}\right)\min_{|z|=1}|p(z)|\Big\}$$

which finishes the verification of Theorem 1.1.

References

1. A. Aziz, Inequalities for the polar derivative of a polynomial, J. Approx. Theory, 55(1988), 183-193.
2. A. Aziz and N.A. Rather, A refinement of a Theorem of Paul Turan concerning polynomials. J. Math. Ineq. Appl., 1(1998), 23 1-238.
3. A. Aziz and W.M. Shah, Some inequalities for the polar derivative of a polynomial. Indian Acad. Sci. (Math. Sci.), 107(1997), 263-270.
4. A. Aziz and W.M. Shah, Inequalities for the polar derivative of a polynomial. Indian J. Pure. Appl. Math., 29(1998), 163-173.
5. N.K. Govil, Some inequalities for derivatives of polynomials, **J.** Approx. Theory, 66(1991), 29-35.
6. M.A. Malik, An integral mean estimate for polynomials. Proc. Amer. Math. Soc., 91(1984), 281-284.
7. D. Mendeleev, Investigations of aqueous solutions based on specific *gravitv,* St. Petersburg, 1887(Russian).
8. W.M. Shah, A generalization of a Theorem of Paul Turan, J. Ramanujan Math. Soc., 1(1996), 67-72.