A Study of Solutions of Variational Inequality, System and Mixed Equilibrium Problem

[1]Sonam Ahuja and [2]Dr. Ravita Bhardwaj

[1]*Research Scholar, Shinhania University, Jhunjhunu, Rajasthan*
[2]*Assistant Professor, Department Of Mathematics, Shinhania University, Jhunjhunu, Rajasthan*

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ABSTRACT

The equilibrium problem incorporates numerous mathematical problems as specific cases for cases, scientific programming problems, complementary problems, variational disparity problems, and saddle point problems, Nash equilibrium problems in non-cooperative games, mini-max imbalance problems, minimization problems and fixed point problems.As of late, much consideration has been given for creating efficient and implementable iterative methods including projection method and its variation frames, extra-gradient method, linear approximation, auxiliary principle method, descent and Newton methods for VIPs and EPs.Then again, numerous issues emerging in different zones of mathematics, for example, optimization, variational analysis and differential equations, can be demonstrated by fixed point issue (FPP) comprising of discovering x ∈C to such an extent that x = Tx, where T is a nonlinear self-guide on C. Fix(T) signifies the arrangement of fixed purposes of T . The investigation of iterative strategies, viz. Helpern iterative strategy, Mann and Ishikawa iterative strategies, and consistency estimate technique for FPP has yielded a large group of works in the most recent decades.There is a substantial number of iterative strategies for concentrate independently, the fixed point issues for nonlinear mappings, variational inequalities, and equilibrium issues. It is of further enthusiasm to create and ponder the iterative techniques to estimate the common arrangements of these issues.

1. Introduction

In 2008, Ceng *et al.* introduced the relaxed extra-gradient method for approximate a common solution of system of variational inequality problems (SVIP): Find (x, y) ∈ C × C such that

$$\begin{cases} \langle \mu_1 B_1 y + x - y, z - x \rangle \geq 0, \quad \forall z \in C, \\ \langle \mu_2 B_2 x + y - x, z - y \rangle \geq 0, \quad \forall z \in C, \end{cases} \qquad (1)$$

where, for every I = 1, 2, $\mu_I > 0$ and $B_i : C \to C$ is a nonlinear mapping, C is a nonempty, shut and convex subset of Hilbert space H; and settled point issue (FPP) for a non-expansive mapping. Facilitate Yao et al. extended the iterative technique given in for SVIP (1) and FPP for a pseudo contractive mapping. Recently extended the methods given in to approximate a typical arrangement of SVIP; variational inequality issue (VIP): Find x ∈ C with the end goal that

$$\langle Dx, y - x \rangle \geq 0, \ \forall y \in C, \qquad (2)$$

Where D: C → H is a nonlinear mapping; and FPP for a prophylactic mapping. Next, we consider blended equilibrium issue (MEP): Find x ∈ C with the end goal that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C, \qquad (3)$$

Where F: C × C → R is bifunction and A: C → H is a nonlinear mapping. If A = 0,
Then MEP (3) is reduced to the equilibrium problem (EP): Find x ∈ C such that

$$F(x, y) \geq 0, \quad \forall y \in C, \qquad (4)$$

In 2008, Moudafi extended Mann sort iterative method to approximate a typical solution of MEP (3) and FPP for a non-expansive mapping. He demonstrated some powerless convergence theorems for the sequences generated by the proposed iterative method. Hide there, Takahashi and Takahashi extended crafted by Moudafi to an Ishikawa sort iterative method to approximate a typical solution of MEP (3) and FPP for a non-expansive mapping. Recently, Yao et al. extended the iterative methods given in to the viscosity approximation method.

Roused by crafted by Ceng et al. , Moudafi , Takahashi and Takahashi , Yao et al. and by the current work going toward this path, we consolidate the casual extra-gradient method with viscosity approximation method to presented another iterative method for approximating a typical solution of VIP(2), SVIP(1), MEP(3) and FPP for an entirely pseudo contractive mapping in a genuine Hilbert space. We establish a solid meeting theorem for the sequences generated by the proposed iterative method. Further, we get a few consequences from the solid union theorem. The results presented here broaden and generalize.

Preliminaries

We recall a few results identified with SVIP (1) and EP (4) which are needed in the sequel. Initially, we have the accompanying technical lemma which is the settled point formulation of SVIP (1):

Lemma2.2.1. For *any* $(x^*, y^*) \in C \times C$, (x^*, y^*) *is a solution of* SVIP (2.1.1) *ifand only if* x^**is a fixed point of the mapping* $Q: C \to C$ *defined by*

$$Q(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \qquad (1)$$

Where $y^* = P_C(x^* - \mu_2 B_2 x^*)$, $\mu_i \in (0, 2\beta_i)$ *and* B_i*:* $C \to H$ *is a* β_i*-inverse strongly monotone mapping for each* $i = 1,2$.

Next, we have the following assumption:

Assumption1. Let $F: C \times C \to R$ *be a bifunction satisfying the followingassumptions:*

(i) $F(x, x) = 0, \quad \forall_x \in C$;

(ii) F *is monotone, i.e.,* $F(x, y) + F(y, x) \le 0, \quad \forall_x \in C$;

(iii) F *is upper hemi-continuous, i.e., for each* x, y, z∈C,

$$\limsup_{t \to 0} F(tz + (1 - t)x, y) \le F(x, y);$$

(iv) For each x∈C, y→F(x, y) *is convex and lower semi-continuous.*

We consider an auxiliary problem related to EP (4): Let $r > 0$ and $x \in H$, find $z \in C$ such that

$$F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C.$$

The following lemma give the properties of solution set Sol (EP (4)) of EP(4).

Lemma 2.*Assume that* $F: C \times C \to R$ *satisfies Assumption* 2. *For* $r > 0$ *and for all* x∈H, *define a mapping* $T_r: H \to C$ *as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x \rangle \ge 0, \quad \forall y \in C\}.$$

Then the following hold:

i. $T_r(x)$ *is nonempty for each* $x \in H$*;*

ii. T_r*is single-valued;*

iii. T_r*is firmly non-expansive, i.e.,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y \rangle, \quad \forall x, y \in H;$$

iv. $Fix(T_r) = Sol(EP(2.1.4))$;

v. Sol (EP (4)) *is closed and convex.*

Iterative method

We demonstrate a solid convergence theorem in light of viscosity approximation and loose additional inclination method for computing an estimated basic arrangement of VIP (2), SVIP (1), MEP (3) and FPP for an entirely pseudo contractive mapping in a genuine Hilbert space.

Theorem1. Give C a chance to be a nonempty, shut and convex subset of a genuine Hilbert space H. For every i = 1, 2, let A, D, B_T: C → H be θ, α, βi-converse emphatically monotone mappings, respectively. Let F : C × C → R be a bi work satisfying the Assumption 1 and T : C → C be a k-strict pseudo contractive mapping with the end goal that θ : = Fix(T) ∩ Sol(SVIP(1)) ∩ Sol(MEP(3)) ∩ Sol(VIP(2.1.2)) 6= ∅. Give f a chance to be a ρ-compression mapping with $\rho \in [0,\frac{1}{2})$. For a given $x_0 \in C$ arbitrarily, let the iterative successions $\{u_n\}$, $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be generated by

$$\begin{cases} F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ z_n = P_C(u_n - \lambda_n D u_n), \\ y_n = \alpha_n f(x_n) + (1-\alpha_n)P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \beta_n x_n + \gamma_n y_n + \delta_n T y_n, \end{cases} \tag{1}$$

Where $\mu_I \in (0,2\beta_I)$, *for each* i= 1,2, $\{r_n\} \subset (0,2\theta)$, $\{\lambda_n\} \subset (0,2\alpha)$, *and* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *are the sequences in* (0, 1) *satisfying the following conditions:*

(i) $\beta_n + \gamma_n + \delta_n = 1$ *and* $(\gamma_n + \delta_n)k \leq \gamma_n$, *for all* $n \geq 0$;

(ii) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ *and* $\liminf_{n\to\infty} \delta_n > 0$;

(iv) $\liminf_{n\to\infty} r_n > 0$, $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;

(v) $\lim_{n\to\infty} \left(\frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n}\right) = 0$;

(vi) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$ *and* $\lim_{n\to\infty} | \lambda_{n+1} - \lambda_n | = 0$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in \theta$ *where* $z = P_\theta f(z)$.

Proof. First, we show that the mapping $(I - r_n A)$ is non-expansive. For anyx, y∈C,

$$\begin{aligned} \|(I - r_n A)x - (I - r_n A)y\|^2 &= \|(x-y) - r_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2r_n\langle x-y, Ax-Ay\rangle + r_n^2\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2 - r_n(2\theta - r_n)\|Ax-Ay\|^2 \\ &\leq \|x-y\|^2. \end{aligned} \tag{2}$$

Similarly, we can show that the mappings $(I - \lambda_n D)$ and $(I - \mu_i B_i)$ are non-expansive For each i = 1, 2. It follows from Lemma 2.2.2 that, $u_n = T_{r_n}(x_n - r_n A x_n)$. Let $x^* \in \theta$

We have $x^* = T_{r_n}(x^* - r_nAx^*)$. Now, we estimate

$$\begin{aligned} \|u_n - x^*\|^2 &= \|T_{r_n}(x_n - r_nAx_n) - T_{r_n}(x^* - r_nAx^*)\|^2 \\ &\leq \|(x_n - r_nAx_n) - (x^* - r_nAx^*)\|^2 \\ &= \|(x_n - x^*) - r_n(Ax_n - Ax^*)\|^2 \\ &\leq \|x_n - x^*\|^2 + r_n^2\|Ax_n - Ax^*\|^2 - 2r_n\langle x_n - x^*, Ax_n - Ax^*\rangle \\ &\leq \|x_n - x^*\|^2 + r_n^2\|Ax_n - Ax^*\|^2 - 2r_n\theta\|Ax_n - Ax^*\| \\ &\leq \|x_n - x^*\|^2 - r_n(2\theta - r_n)\|Ax_n - Ax^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \quad (3)$$

Since $x^*\in\theta$, we have

$$x^* = P_C[P_C(x^* - \mu_2B_2x^*) - \mu_1B_1P_C(x^* - \mu_2B_2x^*)].$$

Putting

$$y^* = P_C(x^* - \mu_2B_2x^*),$$

We see that

$$x^* = P_C(y^* - \mu_1B_1y^*). \quad (4)$$

Since the mapping $D : C \to H$ is α-inverse strongly monotone, we have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|P_C(u_n - \lambda_nDu_n) - P_C(x^* - \lambda_nDx^*)\|^2 \\ &\leq \|(u_n - \lambda_nDu_n) - (x^* - \lambda_nDx^*)\|^2 \\ &\leq \|(u_n - x^*) - \lambda_n(Du_n - Dx^*)\|^2 \\ &\leq \|u_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Du_n - Dx^*\|^2 \\ &\leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2. \end{aligned} \quad (5)$$

Setting $t_n := P_C[P_C(z_n - \mu_2B_2z_n) - \mu_1B_1P_C(z_n - \mu_2B_2z_n)]$ and $v_n := P_C(z_n - \mu_2B_2z_n)$. It follows that

$$\begin{aligned} \|v_n - y^*\|^2 &= \|P_C(z_n - \mu_2B_2z_n) - P_C(x^* - \mu_2B_2x^*)\|^2 \\ &\leq \|(z_n - \mu_2B_2z_n) - (x^* - \mu_2B_2x^*)\|^2 \\ &\leq \|z_n - x^*\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2z_n - B_2x^*\|^2 \\ &\leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \end{aligned} \quad (6)$$

Further, we have

$$
\begin{aligned}
\|t_n - x^*\|^2 &= \|P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] \\
&\quad - P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] \\
&\quad -[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|[P_C(z_n - \mu_2 B_2 z_n) - P_C(x^* - \mu_2 B_2 x^*)] \\
&\quad -\mu_1[B_1 P_C(z_n - \mu_2 B_2 z_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)]\|^2 \\
&\leq \|P_C(z_n - \mu_2 B_2 z_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad -\mu_1(2\beta_1 - \mu_1)\|B_1 P_C(z_n - \mu_2 B_2 z_n) - B_1 P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\leq \|(z_n - \mu_2 B_2 z_n) - (x^* - \mu_2 B_2 x^*)\|^2 \\
&\quad -\mu_1(2\beta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\
&\leq \|z_n - x^*\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 z_n - B_2 x^*\|^2 \\
&\quad -\mu_1(2\beta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\
&\leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \qquad (7, 8)
\end{aligned}
$$

Next, we estimate

$$
\begin{aligned}
\|y_n - x^*\| &= \|\alpha_n(f(x_n) - x^*) + (1-\alpha_n)(t_n - x^*)\| \\
&\leq \alpha_n\|f(x_n) - x^*\| + (1-\alpha_n)\|t_n - x^*\| \\
&\leq \alpha_n(\rho\|x_n - x^*\| + \|f(x^*) - x^*\|) + (1-\alpha_n)\|x_n - x^*\| \\
&= [1-(1-\rho)\alpha_n]\|x_n - x^*\| + (1-\rho)\alpha_n \frac{\|f(x^*) - x^*\|}{1-\rho} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\rho}\right\}.
\end{aligned}
$$

(9)

Since $(\gamma_n + \delta_n)\, k \leq \gamma_n$ for all $n \geq 0$, utilizing Lemma 1. we have

$$
\begin{aligned}
\|x_{n+1} - x^*\| &= \|\beta_n(x_n - x^*) + \gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\| \\
&\leq \beta_n\|x_n - x^*\| + \|\gamma_n(y_n - x^*) + \delta_n(Ty_n - x^*)\| \\
&\leq \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\|y_n - x^*\| \\
&\leq \beta_n\|x_n - x^*\| + (\gamma_n + \delta_n)\max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\rho}\right\} \\
&\leq \max\left\{\|x_n - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\rho}\right\}. \qquad (
\end{aligned}
$$

(10)

By induction on n, we obtain every $\|x_n - x^*\| \leq \max\left\{\|x_0 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\rho}\right\}$ for every $n \geq 0$ and $x_0 \in C$.

Subsequently $\{x_n\}$ is bounded and consequently, we reason that $\{u_n\}$, $\{y_n\}$, $\{z_n\}$, $\{v_n\}$ and $\{t_n\}$ are bounded. Then again, from the non-expansivity of the mapping $(I-\lambda_n D)$, we have

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|P_C(u_{n+1}-\lambda_{n+1}Du_{n+1})-P_C(u_n-\lambda_n Du_n)\|\\ &\leq \|(u_{n+1}-\lambda_{n+1}Du_{n+1})-(u_n-\lambda_n Du_n)\|\\ &= \|(u_{n+1}-u_n)-\lambda_{n+1}(Du_{n+1}-Du_n)+(\lambda_{n+1}-\lambda_n)Du_n\|\\ &\leq \|(u_{n+1}-u_n)-\lambda_{n+1}(Du_{n+1}-Du_n)\|+|\lambda_{n+1}-\lambda_n|\|Du_n\|\\ &\leq \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Du_n\|.\end{aligned} \tag{11}$$

We next estimate

$$\begin{aligned}\|t_{n+1}-t_n\|^2 &= \|P_C[P_C(z_{n+1}-\mu_2B_2z_{n+1})-\mu_1B_1P_C(z_{n+1}-\mu_2B_2z_{n+1})]\\ &\quad -P_C[P_C(z_n-\mu_2B_2z_n)-\mu_1B_1P_C(z_n-\mu_2B_2z_n)]\|^2\\ &\leq \|[P_C(z_{n+1}-\mu_2B_2z_{n+1})-\mu_1B_1P_C(z_{n+1}-\mu_2B_2z_{n+1})]\\ &\quad -[P_C(z_n-\mu_2B_2z_n)-\mu_1B_1P_C(z_n-\mu_2B_2z_n)]\|^2\\ &\leq \|[P_C(z_{n+1}-\mu_2B_2z_{n+1})-P_C(z_n-\mu_2B_2z_n)]\\ &\quad -\mu_1[B_1P_C(z_{n+1}-\mu_2B_2z_{n+1})-B_1P_C(z_n-\mu_2B_2z_n)]\|^2\\ &\leq \|[P_C(z_{n+1}-\mu_2B_2z_{n+1})-P_C(z_n-\mu_2B_2z_n)]\|^2\\ &\quad -\mu_1(2\beta_1-\mu_1)\|B_1P_C(z_{n+1}-\mu_2B_2z_{n+1})-B_1P_C(z_n-\mu_2B_2z_n)\|^2\\ &\leq \|P_C(z_{n+1}-\mu_2B_2z_{n+1})-P_C(z_n-\mu_2B_2z_n)\|^2\\ &\leq \|(z_{n+1}-z_n)-\mu_2(B_2z_{n+1}-B_2z_n)\|^2\\ &\leq \|z_{n+1}-z_n\|^2-\mu_2(2\beta_2-\mu_2)\|B_2z_{n+1}-B_2z_n\|^2\\ &\leq \|z_{n+1}-z_n\|^2.\end{aligned} \tag{12}$$

From (11) and (12), we have

$$\|t_{n+1}-t_n\| \leq \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Du_n\|. \tag{13}$$

We observe that

$$\begin{aligned}\|y_{n+1}-y_n\|^2 &= \|t_{n+1}-\alpha_{n+1}[f(x_{n+1})-t_{n+1}]-t_n-\alpha_n[f(x_n)-t_n]\|\\ &\leq \|t_{n+1}-t_n\|+\alpha_{n+1}\|f(x_{n+1})-t_{n+1}\|+\alpha_n\|f(x_n)-t_n\|\\ &\leq \|u_{n+1}-u_n\|+|\lambda_{n+1}-\lambda_n|\|Du_n\|+\alpha_{n+1}\|f(x_{n+1})-t_{n+1}\|\\ &\quad +\alpha_n\|f(x_n)-t_n\|.\end{aligned} \tag{14}$$

On the other hand $u_n = T_{r_n}(x_n - r_nAx_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1}-r_{n+1}Ax_{n+1})$, we have

$$F(u_n, y) + \langle Ax_n, y - u_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \tag{15}$$

And

$$F(u_{n+1}, y) + \langle Ax_{n+1}, y - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \ \forall y \in C. \tag{16}$$

Take $y = u_{n+1}$ in (15) and $y = u_n$ in (16), we have

$$F(u_n, u_{n+1}) + \langle Ax_n, u_{n+1} - u_n\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0 \tag{17}$$

And

$$F(u_{n+1}, u_n) + \langle Ax_{n+1}, u_n - u_{n+1}\rangle + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0. \tag{18}$$

Adding inequalities (17) and (18), we have the resultant inequality, after using the monotonicity of F,

$$\langle Ax_{n+1} - Ax_n, u_n - u_{n+1}\rangle + \left\langle u_n - u_{n+1}, \frac{u_{n+1} - x_{n+1}}{r_{n+1}} - \frac{u_n - x_n}{r_n}\right\rangle \geq 0,$$

Which implies that

$$\begin{aligned}
0 &\leq \langle u_n - u_{n+1}, r_n(Ax_{n+1} - Ax_n) + \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) - (u_n - x_n)\rangle \\
&\leq \langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right) u_{n+1} + (x_{n+1} - r_n Ax_{n+1}) \\
&\quad -(x_n - r_n Ax_n) - x_{n+1} + \frac{r_n}{r_{n+1}} x_{n+1}\rangle \\
&= \langle u_{n+1} - u_n, u_n - u_{n+1} + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \\
&\quad +(x_{n+1} - r_n Ax_{n+1}) - (x_n - r_n Ax_n)\rangle \\
\|u_{n+1} - u_n\|^2 &\leq \|u_{n+1} - u_n\| \left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\|\right\}
\end{aligned}$$

And hence

$$\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\|.
\end{aligned}$$

Without loss of generality, let us assume that there exists a genuine number c to such an extent that $r_n > c > 0$, for every positive whole number n. At that point the previous inequality implies.

CONCLUSION

This study is to create and consider some iterative techniques for approximating the normal arrangements of variational inequality issues, arrangement of variational disparities, split equilibrium issues, split variational consideration issues and fixed point issues for a (family of) nonlinear mapping(s). we build up an Iterative technique in view of relaxed extra-gradient and viscosity approximation strategies for approximating a common arrangement of VIP(1), SVIP(2), MEP(4) and FPP for an entirely pseudo contractive mapping in a genuine Hilbert space. We build up a strong convergence theorem for the grouping produced by the proposed iterative strategy. Further, we get a few results from the strong convergence theorem. As of late, much consideration has been given for creating efficient and implementable iterative methods including projection method and its variation frames, extra-gradient method, linear approximation, auxiliary principle method, descent and Newton methods for VIPs and EPs.Then again, numerous issues emerging in different zones of mathematics, for example, optimization, variational analysis and differential equations, can be demonstrated by fixed point issue (FPP) comprising of discovering $x \in C$ to such an extent that $x = T_x$, where T is a nonlinear self-guide on C. Fix(T) signifies the arrangement of fixed purposes of T .

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