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The Concept of Fuzzy Measure of Fuzzy Events

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ABSTRACT

In this paper we discuss the concept of fuzzy measure of fuzzy event by using a general form of Fuzzy integral purposed by Murofushi.

1. Introduction

The concept of fuzzy integral with respect to fuzzy measure proposed in the early 1970s by Sugeno, using min-max operator. Later several authors tried to generalize or modify the original definition, aiming at different purposes. Kruse and Weber defined fuzzy integrals w.r.t. decomposable measures, replacing min-max operators by product like operators and Archimedean t-co-norms similar attempts were made by Sugeno and Murofushi using pseudo-addition and multiplication. In order to be coherent with the Lebesgue integral. Murofishi and Sugeno proposed the so-called Choquct integral. They have introduced the general concept of fuzzy t-co-norms integral which can encompass the original definition. Unfortunately, in the very beginning of the history of fuzzy set theory, Zadeh defined the probability of fuzzy event by a Lebesgue integral.

Section I

1.01 Fuzzy Measure

Let x be a non empty set and K be an σ algebra defined on x. A fuzzy measure μ defined on the measurable space (x k) is a set function

$\mu : k \to [0.1]$ satisfying in the following axioms

(i) $\mu(Q) = 0, \mu(X) = 1$

(ii) $A \leq B \Rightarrow \mu(A) \leq \mu(B)$

(x, k.μ) in called fuzzy measure space.

1.02 DEFINITION

Let $\perp$ be a t-co norm and a fuzzy measure satisfying the continunity axion for every increasing sequence, μ is said to be $\perp$ decomposable. If it has the following property.

$$\mu(A \cup B) = \mu(A) \perp \mu(B) \text{ Whenever } A \cap B = \phi$$

A possibility measure is V decomposable measure and probability measure is a decomposable measure.

1.03 DEFINITION

Let (X, K) and (Y, K^1) be two measureable space. A mapping H : $K \to K^1$ is said to be are interpreter for the measureable sets iff. It satisfies.

(i) $H(\phi) = \phi H(x) = y$

(ii) $A \subseteq B \Rightarrow H(A) \subseteq H(B)$

Let (X, K, μ) be a fuzzy measure space. A quadruplet (Y, K^1, M) is called a representation of (X, K, μ) iff it satisfies:

(i) H is interpreter k to Y

(ii) m is an additive measure on (Y, K^1)

(iii) μ = m

SIMPLE FUNCTION

1.04 DEFINITION

A simple function $f : x \to [0,a]$ is such that for all x. $f(x) = \sum_{i=1}^{n} ai\ ID\ (x)$ where Di's are a disjoint family to subsets of x $I_{Di}(x)$

denote characteristic function of Di. We assume $0 \le a_1 \le a_2 \le .... \le a_n, \le a$

FUZZY T-CO-NORM INTEGRALS

1.05 DEFINITION

The Sugeno integral of a measure $f : x \to [0,1]$ is defined by f to $\mu a\varepsilon[0,1][a \wedge \mu(x : f(x) > q]$ for simple functions, the expression reduces to fo $\mu\ V_{i=1}^{n}$ (ai μ(Ai))

1.06 DEFINITION

The choqut integral of measurable function $F : X \to [0,1]$ is defined by (c) $\int d\mu = \int_0^1 \mu(\{1f(x)7a)da\mu(\{x : f(x) > a)da$

For Simple function, Choquet integral is defined as

(c) $\int f\ d\mu = \Sigma_{i=1}^{n}(ai - a_{i-1})\mu(Ai)$

1.07 DEFINITION

$(\Delta, \perp \underline{\perp}\ \diamondsuit)$ is a t-co norm system for integration iff

(i) $\diamondsuit \perp \underline{\perp}$ are continuous which is V or Archimedean

(ii) $\diamondsuit : ([0,1] \times [0,1], \perp) \to ([0,1], \underline{\perp})$ is product like operation

SATISFYING

(a) $\diamondsuit$ is continuous on $[0, 1]^2$

(b) a $\diamondsuit$ x0 $\Leftrightarrow$ a = 0 or x = 0

(c) when $a \Delta b < 1.(a\Delta b)$ $\diamondsuit$ x = (a $\diamondsuit$ x)

$\underline{\perp}$ (b $\diamondsuit$ x), $\forall x \in [0,1]$

(d) When $a\Delta b < 1, (a\Delta b)$ $\diamondsuit$ x = (a $\diamondsuit$ x)

$\underline{\perp}$ (b $\diamondsuit$ x), $\forall x \in [0,1]$

When ([0, 1], Δ), {[0, 1], $\perp$) and ([0, 1], $\underline{\perp}$) are respectively : the spaces of values of integrand and, measure and integral.

When all three t-co norms and Archimedean the system is called Archimedean. Let us denote by h, k, g the generator of $\Delta \perp, \underline{\perp}$ respectively

then following property holds :

$\forall a > 0, \forall x > 0$

or $\forall a \in [0,1] \forall x \in [0,1]$

a $\diamondsuit$ x = h^{-1} [h(1) ^ (k/a). g(x)]

1.08 DEFINITION

For a given t-conorm Δ. We define a operator Δ, on $[0, 1]^2$

By $a^{-}\Delta b = \inf(c : b\Delta c >, a)$

When Δ = V we have

$$a^{-}\ \Delta b = \begin{cases} a \text{ if } a > b \\ 0 \text{ if } a < b \end{cases}$$

1.09 DEFINITION

Let (X, K, μ) be a fuzzy measure space and $f = (\Delta, \perp, \underline{\perp}, \diamondsuit)$ a t-co norm system. For a measureable simple function $f : x \to [0,1]$, the fuzzy t-co-norm integral of f based on f w-r-t. μ is defined by

$(f)\int f \diamondsuit d\mu\ \underline{\perp}_{i=I}^{n}(a, \bar{\wedge} a_{i-1}) \diamondsuit \mu(Av)$

INTEGRAL WITH RESPECT TO DUAL MEASURE

1-Dual measure

Usually the notion of dual measure refers to relation holding between possibility and necessity, or belief and plausibility function, a pair of measures.

$(\mu,\mu^*)\mu^*(.) = 1\text{-}\mu(.c)$ or $\mu^*(.) = \mu(x) - \mu(.c)$ for generality, where C denote the compliment. Note that probability measures are self dual measures

1.10. DEFINITION

Let $\perp$ be an Archimedean nilpotent t-co-norm (μ,μ) is a pair of $\perp$ dual measures iff $\forall A \in X, \mu^{\perp}(A) = 1{:}\perp_{\mu}(A^{C})$

1.11 DEFINITION

A fuzzy t-co-norm integeral is a Choquet like integral iff $(\Delta,\perp,\underline{\perp},0)$ satisfies

(i) a$\diamondsuit$ $x = 1 \Leftrightarrow a = 1$ and $x = 1$

(ii) $\Delta,\perp$ and $\perp$ are Archimedean nilpotent t-co-norms.

Now we shall prove the following lemma concering Choquet integral :

Lemma 1 :

Let f be a measureable functions s to for all

$x \in X, 0 \le f(x) \le a$ and $\mu : k \to [0,\mu(X)]$ a fuzzy measure whose dual measure μ^* is defined by $\mu^*(.) = \mu(x) - (.c)$ Then the following property holds

(C) $\int f[a - f(x)] = a\mu(x) - (c)\int\ fd\mu^*$

PROOF

In continuous case we use directly the definition of choquet integral expanding the left side of equation (i) gives

$$(c)\int f[a - f(x)]d\mu$$

$$= \int_0^a \mu(x / a - f(x) > r)dr$$

$$= \int_0^a \mu(\{x / f(x \le S)\}ds$$

with S : a – r we now expand, the right side.

$$a\mu(x) - (c)\int f\, d\mu^* = a\mu(x) = \int_0^a \mu^*\{(x : gx > s)\}ds$$

$$= \partial\mu(x) = \int_0^a [\mu(x) - \mu(\{x : f(x) > s\})ds$$

$$= a\mu(x) - a\mu(x) + \int_0^a \mu(\{x : f(x) > s)\}ds$$

$$= \int_0^a \mu(\{x : f(x) \le s)\})ds$$

Where we have used the fact

$$(c)\int fd\mu = \int_0^a \mu(\{f \ge r)dr = \int_0^a \mu(\{f > r\})dr$$

Choquet like integral can be expressed using the generator function. So we prove the following lemma.

Lemma 2

Let $f = (\Delta,\perp,\underline{\perp},\diamondsuit)$ define a choquet like-integral, and f : x – [0, 1] be a measurable function. When the following hold :

Condition (i) of definition 1 : 13 is equivalent to h (1) = k (1), g (1)---(α)

(i) $(f)\int f \diamondsuit d\mu = h^{-1}$ {(c) $\int$ kofd (goμ)]-------(β)

From (i) of definition 1.13 $\Rightarrow(\alpha)$

By |$\diamondsuit$| = 1 and from the equation $\forall\, a \in [0,1]$, $\forall\, x \in [0,1]$

a$\diamondsuit$ x h^{-1} [h (1) x (k (a), g(x)]

We get h (1) $\le$ k (1) – g(1)

Suppose that h (1) < k (1). g (1), then $\frac{h(1)}{g(1)}$ lies in the interval]0 = k(0), k(1)[and hence using the continuity of k and intermediate value theorem, there exists 0 < a < 1 s.t.

$k(a) = \text{i.e.} \frac{h(1)}{g(1)} \text{i.e.}\ a \Diamond 1 = 1$

Which contradicts the hypothesis then h (1) = k (1). g (1)

Equation $\alpha \Rightarrow (i)$ suppose now a $\Diamond$z = 1 then h (1) = h(i) ^ k(a).g(a) and become k, g are strictly increasing, it follows the hypothesis that a = 1 and x = 1

(ii) Because of the monotonocity of k_1 k $\Diamond$ $\varphi >, kof$

Also from the monofonocity of the choquet integral it follows that

$(c) \int k \Diamond f\, d(g \Diamond \mu) \leq (c) \int k \Diamond f\, (go\mu)$

$= k\,(1)\, g(\mu(X))$

$= k\,(1)\, g(1) = h(1)$ using

(i) above thus the equation

$(f) \int \Diamond d\,\mu = h^{-1} [h\,(1) \wedge (c) \int k \Diamond f\, d\,(go\mu)$

reduces to equation

$(f) \Diamond d\mu = h^{-1} [(c) \int k0fd(g0\mu)]$

Now using the two lemma we can show the following result.

Theorem 1

Let $f = (\Delta, \perp, \underline{\perp} \Diamond)$ define a choquet integral and f : x – [0, 1] be measurable function, then for every pair $(\mu, \mu^{\perp})$ of $\perp -$ dual measures the following holds.

$(f) \int f \Diamond d\,\mu = 1 - \underline{\perp} (f) \int (1 - \Delta^r) \Diamond d\mu^1$ ……….(r)

PROOF

We denote by different operators, associated to t = corm, $\Delta, \perp, \underline{\perp}$ respectively, whose generators are k, g, h respectively. We first need the following result :

$(go\,\mu^{\perp}) * (A) = go\,\mu$

$\mu * c.1 \underline{\underline{\Delta}} \mu(x) - \mu(.c)$

for every A in K we have

$go\ \mu^{\perp}(A) = go\mu^{\perp}(x) - go\mu^{\perp}(f)$

$= g(1) - g[1 - \perp \mu(A)]$

$= g(1) - [ov / g(1) - go\mu(A)]$

$= g(1) - g(1) + go(A)$

$= go(A)$

Now we shall prove the main theorem : Expanding the right side of equation (r) and using lemma 1, 2 (ii) and the above results gives :

$1 - \underline{\perp}(f) \int (1 - \Delta f) \Diamond d\mu^{\perp} \perp)$

$= h^{-1} \int ovh(1) - (c) \int ko1 - kof) d(go\mu$

$= h^{-1} \{h(1) - k(1)\{go\,\mu^{\perp})(x) + (c) \int k0fd)(g0\mu^{\perp})]$

$= h - 1[h(1) - k(1)(g(1) + (c) \int k\,o\,f\, d(go\mu)$

$= h^{-1}(c) \int kofd(gp\mu)] \leq\leq$

$= (f) f \Diamond d\,\mu$

SECTION – 2

Now we discuss the fuzzy measure of fuzzy event.

In 1968 Zadeh defined the concept of probability of fuzzy event $\bar{A}$ in order to extend to fuzzy sets the fundamental concept of probability of an event in classical probability theory. The definition is $P(\bar{A}) = \int \varphi\bar{A}(x)dp \to \partial$ were Qa, denote the membership function of $\bar{A}$ and P a probability.

Let us denot $\bar{\mu}$ an extension of the fuzzy measure μ for measureable fuzzy sets. Inventively, the definition of $\bar{\mu}$ munsatisfy the following properties.

(i) $\bar{\mu}$ must be an extension of μ on P(X) i.e. $\bar{\mu}$ (A) = μ (A) wherever $A \in k$,

(ii) $\bar{A} \subseteq B \Rightarrow \mu(A) \leq \mu(B)$ (monot.)

(iii) If μ is continues for every increasing sequence (or decreasing) of measurable sub set. Then for every increasing for decreasing) sequence of fuzzy sub sets.

$\overline{\{An\}}$ we have $\bar{\mu}(\mathrm{Lim}_{n-\alpha} \overline{An}) = \lim_{n-\alpha} \mu(An) \lim_{n-\alpha} \mu$ An (Continuity)

(iv) For every fuzzy measure on P(X) a dual measure exists. Using a system of Archimedean nicpotent t-conorms the most general definition is given by

$$\bar{\mu}\underline{\perp} = 1 - \perp \bar{\mu}\bar{A}^{\,c}\Delta$$

Where $\bar{A}^{C\Delta}$ is the Δ-negation of $\bar{A}$ defined by $Q_{\bar{A}} c^{\Delta}, 1 - \Delta Q_{\bar{A}}(x)$

and $\mu^{\perp}$ is an extension of $\mu^{\perp}$ the dual measure of μ.

Theorem 2.01

Let = $(\Delta_f = \perp, \perp, \diamondsuit)$be of the second kind (Chq lest – like intagred) then for even fuzzy measure μ, the fuzzy measure extended on P (x) defined by f satisfies all conditions (i) to (v). However concerning (v), only the case of the probability is recovered.

PROOF

Let us denote by k, g, h, the generator of $\Delta, \perp, \perp$ respectively, with h = k

(i) g. (i) let A be an ordinary subset of X, using the definition we have

$$\mu(A) = \begin{pmatrix}1\\1\end{pmatrix} \int Q_{\Lambda} \diamondsuit\, d\mu = h^{-1}\left[(c)\int \mathrm{Kod}_{\Lambda}{}^{\infty}(go\mu)\right]$$

$$= h^{-1}[k(1).go\mu(A)] = g^{-1}\left[\frac{1}{k(1)} k(1) go\mu(A)\right] = \mu(A)$$

(ii) Let $\bar{A} \leq \bar{B}$ ie $Q_{\bar{A}} \leq Q_{\bar{B}}$ from the monotono city of the Choquet integral and of k, h we obtain the desired result.

(iii) Let $\{\bar{A}_x\}$ be an increasing (respectively decreasing) sequence of fuzzy events and μ be a fuzy measure for every increasing (resp. decreasing) sequence of measurable sets. The Choquet integral possess the following property for every increasing (resp. decreasing) sequence fn $\lim_{n\to\infty}(c)\int fn d\mu = (c)\int \lim_{n\to\infty}(Q_A)(fn)d\mu$

then in particular $\lim_{n\to\infty}(c)\int Q_{\bar{A}} d\mu = \int \lim_{n\to} \int (Q\ \bar{A})d\mu$

thus using the continuity of h, k, we can write.

$$\lim_{n\to\infty}(A_n) = (\lim_{n\to\infty} h^{-1})[(c)\int k(Q_{An})(go\mu)]$$

(iv) Proof of (iv) is clear

Proof of v clear from the definition.

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