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Solving Some Linear & Nonlinear Differential-Difference Equations

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ABSTRACT

Laplace decomposition method is based on Laplace transform method and Adomian decomposition method. In this paper we show that the method is applicable to certain successive interval valued linear as well as nonlinear differential-difference equations of order (1,1), that means the differential is of order one and the difference is of order one. It is also shown that the method gives exact solution for linear problems and suitable approximate solution for nonlinear problems. The example is selected to illustrate the applicability of the method.

1. Introduction

Linear and nonlinear simple differential equations of first order with explicit correct solutions are important test cases for numerical methods[1, 2] as well as analytical methods[3, 4] as they offer practical instructions to develop methods such that solutions can be rendered as precise as possible in the numerical case and solutions can be presented as elegantly as possible in the analytical case. By analyzing stability and convergence, numerical methods get refined. Likewise, analytical approximate solutions are refined with the help of Pade approximation by analytical continuation and asymptotic analysis [5, 6, 7, 8, 9, 10]. Standard programming programs such as MATLAB, MATHEMATICA, MAPLE and so on will carry out all types of approaches as easily as possible. Taking these facts into account, this chapter is devoted to the analysis of two analytical methods, such as the Laplace Transform Method and the Laplace Decomposition Method [11, 12, 13, 14, 15, 16], [17-19], [20, 21, 22, 23, 24], which resolve any linear differential equation with the main idea of making them applicable to the resolution of linear or nonlinear differential equations of order (1,1) with appropriate interval condition.

For the research the following simple types of differential-differential equations[25, 26] are regarded:

1. Linear differential-difference equation of order (1,1) :

$$u^{!}(t) + cu(t-\omega) = f(t), \quad t > \omega, \qquad (1.1)$$
$$u(t) = a + bt, \quad 0 \le t \le \omega,$$

where $c \neq 0$, a and b are real constants, $f(t)$ is a given function of exponential order and ω is a positive difference parameter.

2. Nonlinear differential-difference equation of order (1,1) :

$$u^{!}(t) = f(u(t-\omega)), \quad t > \omega, \qquad (1.2)$$
$$u(t) = a + bt, \quad 0 \le t \le \omega,$$

where a and b are real constants, $f(u)$ is a given nonlinear function of exponential order and ω is a positive difference parameter.

The next section describes the Laplace transform method and the Laplace decomposition method for linear problem (1.1), and describes the Laplace decomposition method for nonlinear problem (1.2). A set of three examples are worked out in the ensuing section. The last section contains concluding remarks and the scope for further work to be done in the next chapters.

2. METHODS TO SOLVE SOME SPECIFIC DIFFERENTIAL-DIFFERENCE EQUATIONS OF ORDER (1,1)

Linear differential-difference equation of order **(1,1)**

Let us consider linear differential-difference equation of order (1,1) given by (2.1):

$$u^{!}(t) + cu(t-\omega) = f(t), \quad t > \omega,$$
$$u(t) = a + bt, \quad 0 \le t \le \omega,$$

where $c \neq 0$, a,b are real constants, $f(t)$ is a given function of exponential order and ω is a positive difference parameter.

First we note that,

$$\int_{\omega}^{\infty} u'(t)\, e^{-st}\, dt = L\{u'(t)\} - \int_{0}^{\omega} b\, e^{-st}\, dt = L\{u'(t)\} + \frac{b}{s}\left(e^{-\omega s} - 1\right).$$

Laplace transform method [26]

Now let us multiply both sides of (2.1) by e^{-st}, $s > 1$ and integrate between ω and ∞, to obtain

$$\int_{\omega}^{\infty} u'(t)\, e^{-st}\, dt \; + \; c\int_{\omega}^{\infty} u(t-\omega)\, e^{-st}\, dt = \int_{\omega}^{\infty} f(t)\, e^{-st}\, dt.$$

Next let us use initial interval condition and apply the formula of Laplace transform for shifting the variables from $(t-\omega)$ to t, we get

$$L\{u'(t)\} + \frac{b}{s}\left(e^{-\omega s} - 1\right) + c\, e^{-\omega s} L\{u(t)\} \;=\; e^{-\omega s} L\{f(t+\omega)\}. \qquad (2.2)$$

After simplifications, finally, we arrive at

$$L\{u(t)\}+c\,\frac{e^{-\omega s}}{s}L\{u(t)\} \;=\; \frac{a}{s}+\frac{b}{s^2}+\left[-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-\omega s}. \qquad (2.3)$$

The expression for $L\{u(t)\}$ can be written as follows:

$$L\{u(t)\} \;=\; \frac{\dfrac{a}{s}+\dfrac{b}{s^2}+\left[-\dfrac{b}{s^2}+\dfrac{L\{f(t+\omega)\}}{s}\right]e^{-\omega s}}{1+\dfrac{c\,e^{-\omega s}}{s}}.$$

Since $s > 1$, we have the following series expansion:

$$\begin{aligned}
L\{u(t)\} &= \left(\left[\frac{a}{s}+\frac{b}{s^2}\right]+\left[-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-\omega s}\right)\\
&\quad\times\left(1-\frac{ce^{-\omega s}}{s}+\frac{c^2e^{-2\omega s}}{s^2}+\cdots+(-1)^n\frac{c^ne^{-n\omega s}}{s^n}+\cdots\right)\\
&= \left[\frac{a}{s}+\frac{b}{s^2}\right]+\left[-\frac{ac}{s^2}-\frac{bc}{s^3}-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-\omega s}\\
&\quad+\frac{-c}{s}\left[-\frac{ac}{s^2}-\frac{bc}{s^3}-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-2\omega s}\\
&\quad+\frac{c^2}{s^2}\left[-\frac{ac}{s^2}-\frac{bc}{s^3}-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-3\omega s}\\
&\quad\vdots\\
&\quad+\frac{(-1)^{n-1}c^{n-1}}{s^{n-1}}\left[-\frac{ac}{s^2}-\frac{bc}{s^3}-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-n\omega s}\\
&\quad\vdots\\
&= \sum_{n=0}^{\infty}L\{U_n(t)\}\,e^{-n\omega s}.
\end{aligned} \qquad (2.4)$$

Therefore by applying inverse laplace transform, we get the desired series expansion for the solution $u(t)$:

$$u(t) \;=\; \sum_{n=0}^{\infty}U_n(t-n\omega)e(t-n\omega),$$

where $e(t-n\omega)$ is a unit step function given by,

$$e(t-n\omega)=\begin{cases}0, & t<n\omega,\\ 1, & t>n\omega.\end{cases}$$

Therefore the exact solution for each interval is given by,

$$u(t) \;=\; \sum_{n=0}^{N}U_n(t-n\omega), \qquad N\omega\le t\le(N+1)\omega, \qquad N=0,1,2,\ldots.$$

Laplace decomposition method

Let us apply Laplace decomposition method for (1.1)

$u'(t) + cu(t-\omega) = f(t), \quad t>\omega,$

with the initial interval condition

$u(t) = a + bt, \quad 0 \le t \le \omega.$

For working out the initial steps of the Laplace transform method, we can directly consider the following equation (2.3) and rewrite as

$$L\{u(t)\} \;=\; \frac{a}{s}+\frac{b}{s^2}+\left[-\frac{b}{s^2}+\frac{L\{f(t+\omega)\}}{s}\right]e^{-\omega s}-c\,\frac{e^{-\omega s}}{s}L\{u(t)\}. \qquad (2.5)$$

Now we seek the following type of decomposition for $L\{u(t)\}$:

$$L\{u(t)\} \;=\; \sum_{n=0}^{\infty}e^{-n\omega s}L\{u_n(t)\}, \qquad (2.6)$$

which may be regarded as Laplace decomposition [27]. Now the main idea of Laplace decomposition is to set an iteration as follows:

$$\sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} = \frac{a}{s} + \frac{b}{s^2} + \left[-\frac{b}{s^2} + \frac{L\{f(t+\omega)\}}{s} - c\,\frac{L\{u_0(t)\}}{s}\right] e^{-\omega s} - \frac{c}{s}\sum_{n=2}^{\infty} e^{-n\omega s} L\{u_{n-1}(t)\}. \quad \ldots\ldots\ldots(2.7)$$

For $n = 0,1,2,...$, equate the co-efficient of $e^{-n\omega s}$ on both sides of (2.7) to get $L\{u_n(t)\}$.

$$L\{u_0(t)\} = \frac{a}{s} + \frac{b}{s^2}.$$

$$L\{u_1(t)\} = -\frac{b}{s^2} + \frac{L\{f(t+\omega)\}}{s} - c\,\frac{L\{u_0(t)\}}{s}.$$

$$L\{u_n(t)\} = -\frac{c}{s} L\{u_{n-1}(t)\}, \text{for } n \geq 2.$$

By applying inverse Laplace transform for the Laplace decomposition series (2.6), we get

$$u(t) = \sum_{n=0}^{\infty} u_n(t-n\omega)e(t-n\omega),$$

where $e(t - n\omega)$ is a unit step function given by,

$$e(t-n\omega) = \begin{cases} 0, & t < n\omega, \\ 1, & t > n\omega. \end{cases}$$

Therefore the exact solution for each interval is given by,

$$u(t) = \sum_{n=0}^{N} u_n(t-n\omega), \qquad N\omega \leq t \leq (N+1)\omega, \qquad N = 0,1,2,\ldots\,.$$

2. Nonlinear differential-difference equation of order **(1, 1)**

Now let us consider the nonlinear differential-difference equation of order (1,1) given by (1.2):

$$u'(t) = f(u(t-\omega)), \quad t > \omega,$$
$$u(t) = a + bt, \quad 0 \leq t \leq \omega,$$

where a,b are real constants, $f(u)$ is a given nonlinear function of exponential order and ω is a positive difference parameter. First we note that, with the help of initial interval condition, we have

$$\int_{\omega}^{\infty} u'(t)\, e^{-st}\, dt = L\{u'(t)\} - \int_0^{\omega} b\, e^{-st}\, dt = L\{u'(t)\} + \frac{b}{s}\left(e^{-\omega s} - 1\right).$$

Laplace decomposition method

Now let us multiply both sides of (2.2) by e^{-st}, $s > 1$ and integrate between ω and ∞, to obtain

$$\int_{\omega}^{\infty} u'(t)\, e^{-st}\, dt = \int_{\omega}^{\infty} f(u(t-\omega))\, e^{-st}\, dt.$$

Let us apply suitable shifting of variables to obtain

$$L\{u'(t)\} + \frac{b}{s}\left(e^{-\omega s} - 1\right) = e^{-\omega s} L\{f(u(t))\}.$$

After applying formula of Laplace transform for first order derivative, we arrive at

(2.8)

$$L\{u(t)\} = \frac{a}{s} + \frac{b}{s^2} - \frac{b\, e^{-\omega s}}{s^2} + \frac{e^{-\omega s}}{s^2} L\{f(u(t))\}.$$

Now we seek the following type of decomposition for $L\{u(t)\}$:

$$L\{u(t)\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\}$$

The decomposition for $L\{f(u(t))\}$ is given by,

$$\begin{aligned}
L\{f(u(t))\} &= L\{f(u_0(t))\} \\
&+ e^{-\omega s} L\left\{\left[\frac{\frac{d}{du}f(u(t))\,|_{u=u_0}}{1!}\right]u_1(t)\right\} \\
&+ e^{-2\omega s} L\left\{\left[\frac{\frac{d}{du}f(u(t))\,|_{u=u_0}}{1!}\right]u_2(t) + \left[\frac{\frac{d^2}{du^2}f(u(t))\,|_{u=u_0}}{2!}\right]u_1^2(t)\right\} \\
&+ e^{-3\omega s} L\left\{\left[\frac{\frac{d}{du}f(u(t))\,|_{u=u_0}}{1!}\right]u_3(t)\right. \\
&\quad \left. + \left[\frac{\frac{d^2}{du^2}f(u(t))\,|_{u=u_0}}{2!}\right]2u_1(t)u_2(t) + \left[\frac{\frac{d^3}{du^3}f(u(t))\,|_{u=u_0}}{3!}\right]u_1^3(t)\right\} \\
&+ \cdots \\
&= \sum_{n=0}^{\infty} e^{-n\omega s} L\{A_n(t)\}.
\end{aligned} \tag{2.9}$$

which may be regarded as Laplace decomposition. In (2.9),

$$\begin{aligned}
A_0(t) &= f(u_0(t)). \\
A_1(t) &= \left[\frac{\frac{d}{du}f(u(t))\,|_{u=u_0}}{1!}\right]u_1(t)
\end{aligned}$$

and in general, for $n \geq 2$, A_n is n^{th} degree Adomian Polynomial [28] of $f(u(t))$ in the powers of $u_1(t), u_2(t), \ldots, u_n(t)$ given by,

$$\begin{aligned}
A_n(t) &= \left[\frac{\frac{d}{du}f(u(t))\,|_{u=u_0}}{1!}\right]u_n(t) \\
&+ \left[\sum_{k=2}^{n}\frac{\frac{d^k}{du^k}f(u(t))\,|_{u=u_0}}{k!}\right]\sum_{i_1+i_2+\cdots+i_k=n} u_{i_1}(t)u_{i_2}(t)\cdots u_{i_k}(t)
\end{aligned}.$$

Now the main idea of Laplace decomposition is to set an iteration as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} e^{-n\omega s} L\{u_n(t)\} &= \frac{a}{s} + \frac{b}{s^2} + \left(-\frac{b}{s^2} + \frac{L\{A_0(t)\}}{s^2}\right)e^{-\omega s} \\
&+ \frac{1}{s^2}\sum_{n=2}^{\infty} e^{-n\omega s} L\{A_{n-1}(t)\}.
\end{aligned} \tag{2.10}$$

One may compute $L\{u_n(t)\}$ iteratively as follows :

$$\begin{aligned}
L\{u_0(t)\} &= \frac{a}{s} + \frac{b}{s^2}. \\
L\{u_1(t)\} &= -\frac{b}{s^2} + \frac{L\{A_0(t)\}}{s^2}. \\
L\{u_n(t)\} &= \frac{1}{s^2}L\{A_{n-1}(t)\}, \qquad n = 2, 3, 4, \ldots .
\end{aligned}$$

By applying inverse Laplace transform for the Laplace decomposition series, we get

$$u(t) = \sum_{n=0}^{\infty} u_n(t - n\omega)e(t - n\omega),$$

where $e(t - n\omega)$ is a unit step function given by,

$$e(t - n\omega) = \begin{cases} 0, & t < n\omega, \\ 1, & t > n\omega. \end{cases}$$

Therefore the approximate solution for each interval is given by,

$$u(t) \approx \sum_{n=0}^{N} u_n(t - n\omega), \qquad N\omega \leq t \leq (N+1)\omega, \qquad N = 0, 1, 2, \ldots . \tag{2.11}$$

EXAMPLE

Let us consider the following standard linear differential-difference equation of retarded type guided by the literature [26]:

$$u'(t) - u(t-\omega) = 1, \qquad t > \omega, \qquad \text{...(2.12)}$$

with the initial interval condition

$$u(t) = 1, \qquad 0 \leq t \leq \omega. \qquad \text{...(2.13)}$$

Laplace transform method [10]

The standard Laplace transform method is quite suitable for solving this problem.

Because, one can express Laplace transform of $u^0(t)$ and $u(t-\omega)$ in terms of $L\{u(t)\}$. For this purpose let us multiply both sides by e^{-st}, $s > 1$ and integrate between ω and ∞, to obtain

$$\begin{aligned}\int_\omega^\infty e^{-st}u'(t)dt &= \int_\omega^\infty 1\ e^{-st}dt + \int_\omega^\infty e^{-st}u(t-\omega)dt \\ L\{u'(t)\} &= \frac{e^{-\omega s}}{s} + e^{-\omega s}L\{u(t)\} \\ sL\{u(t)\} - 1 &= \frac{e^{-\omega s}}{s} + e^{-\omega s}L\{u(t)\}.\end{aligned}$$

The expression for $L\{u(t)\}$ can be written as follows:

$$L\{u(t)\} = \frac{\left(\frac{1}{s} - \frac{e^{-\omega s}}{s^2}\right) + 2\frac{e^{-\omega s}}{s^2}}{1 - \frac{e^{-\omega s}}{s}}.$$

Since $s > 1$, we have the following series expansion:

$$L\{u(t)\} = \frac{1}{s} + 2\frac{e^{-\omega s}}{s^2}\left[1 + \frac{e^{-\omega s}}{s} + \frac{e^{-2\omega s}}{s^2} + \cdots + \frac{e^{-n\omega s}}{s^n} + \cdots\right] = \frac{1}{s} + 2\sum_{n=1}^{\infty}\frac{e^{-n\omega s}}{s^{n+1}} \qquad (2.14)$$

Now by applying inverse laplace transform, we get

$$\begin{aligned}u(t) &= L^{-1}\left\{\frac{1}{s}\right\} + 2\sum_{n=1}^{\infty} L^{-1}\left\{\frac{e^{-n\omega s}}{s^{n+1}}\right\} \\ &= 1 + 2\sum_{n=1}^{\infty}\frac{(t-n\omega)^n}{n!}e(t-n\omega)\end{aligned} \qquad (2.15)$$

where $e(t - n\omega)$ is a unit step function given by,

$$e(t-n\omega) = \begin{cases} 0, & t < n\omega, \\ 1, & t > n\omega. \end{cases}$$

Therefore the exact solution $u(t)$ takes the following form in each interval:

$$u(t) = 1 + 2\sum_{n=1}^{N}\frac{(t-n\omega)^n}{n!}, \qquad N\omega \leq t \leq (N+1)\omega,$$

$$N = 1, 2, 3, \ldots \qquad (2.16)$$

Let us note that $u'(t)$ is continuous except at $t = \omega$.

$$u'(t) = 0, \qquad 0 \leq t \leq \omega,$$

$$u'(t) = 2, \qquad \omega \leq t \leq 2\omega,$$

$$u'(t) = 2 + 2(t-2\omega), \qquad 2\omega \leq t \leq 3\omega,$$

$$\vdots$$

$$u'(t) = 2 + \sum_{n=2}^{N-1}\frac{(t-n\omega)^{n-1}}{(n-1)!}, \qquad (N-1)\omega \leq t \leq N\omega,$$

$$u'(t) = 2 + \sum_{n=2}^{N}\frac{(t-n\omega)^{n-1}}{(n-1)!}, \qquad N\omega \leq t \leq (N+1)\omega,$$

In order to gain continuity at $t = \omega$ for $u'(t)$, we modify the initial interval condition (2.13) and then solve the following problem (2.12): $u(t) = 1 + 2t$, Again let us multiply both sides of (2.1.1) by e^{-st}, $s > 1$ and integrate between ω and ∞, we obtain

$$\begin{aligned}\int_{\omega}^{\infty} e^{-st}u'(t)dt &= \int_{\omega}^{\infty} 1\ e^{-st}dt + \int_{\omega}^{\infty} e^{-st}u(t-\omega)dt\\ \int_{0}^{\infty} e^{-st}u'(t)dt - \int_{0}^{\omega} 2e^{-st}dt &= \frac{e^{-\omega s}}{s} + e^{-\omega s}\int_{0}^{\infty} e^{-st}u(t)dt\\ L\left\{u'(t)\right\} + \frac{2}{s}(e^{-\omega s}-1) &= \frac{e^{-\omega s}}{s} + e^{-\omega s}L\left\{u(t)\right\}\\ sL\left\{u(t)\right\} - 1 &= \frac{2}{s} - \frac{e^{-\omega s}}{s} + e^{-\omega s}L\left\{u(t)\right\}.\end{aligned} \tag{2.18}$$

The expression for $L\{u(t)\}$ can be written as follows:

$$\begin{aligned}L\left\{u(t)\right\} &= \frac{\left(\frac{1}{s}-\frac{e^{-\omega s}}{s^2}\right)+\frac{2}{s^2}}{1-\frac{e^{-\omega s}}{s}}\\ &= \frac{1}{s} + 2\sum_{n=0}^{\infty}\frac{e^{-n\omega s}}{s^{n+2}}.\end{aligned} \tag{2.19}$$

Now by applying inverse laplace transform, we get

$$\begin{aligned}u(t) &= L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s^2}\right\} + 2\sum_{n=1}^{\infty}L^{-1}\left\{\frac{e^{-n\omega s}}{s^{n+2}}\right\}\\ &= 1 + 2t + 2\sum_{n=1}^{\infty}\frac{(t-n\omega)^{n+1}}{(n+1)!}e(t-n\omega)\end{aligned}$$

and Therefore the exact solution $u(t)$ takes the following form in each interval:

$$\begin{aligned}u(t) &= 1 + 2t + 2\sum_{n=1}^{N}\frac{(t-n\omega)^{n+1}}{(n+1)!}, \qquad N\omega \le t \le (N+1)\omega,\\ &\qquad\qquad N = 1, 2, 3, \ldots\ .\end{aligned} \tag{2.20}$$

Let us note that $u'(t)$ is continuous everywhere.

$$\begin{aligned}u'(t) &= 2, & 0 \le t \le \omega,\\ u'(t) &= 2 + 2(t-\omega), & \omega \le t \le 2\omega,\\ u'(t) &= 2 + 2(t-\omega) + 2(t-2\omega), &\\ &\ \vdots &\\ u'(t) &= 2 + \sum_{n=2}^{N-1}\frac{(t-n\omega)^n}{n!}, & (N-1)\omega \le t \le N\omega,\\ u'(t) &= 2 + \sum_{n=2}^{N}\frac{(t-n\omega)^n}{n!}, & N\omega \le t \le (N+1)\omega,\end{aligned}$$

Laplace decomposition method

Let us apply Laplace decomposition method for (2.12)

$$u'(t) - u(t-\omega) = 1, \qquad t > \omega,$$

with the initial interval condition (2.1.16)

$$u(t) = 1 + 2t, \qquad 0 \le t \le \omega.$$

For working out the method, we can directly consider the following equation (2.18) and rewrite as,

$$L\left\{u(t)\right\} = \frac{1}{s} + \frac{2}{s^2} - \frac{e^{-\omega s}}{s^2} + \frac{e^{-\omega s}}{s}L\left\{u(t)\right\}. \tag{2.21}$$

Now we seek the following type of decomposition for $L\{u(t)\}$:

$$L\left\{u(t)\right\} = \sum_{n=0}^{\infty} e^{-n\omega s}L\left\{u_n(t)\right\}, \tag{2.22}$$

which may be regarded as Laplace decomposition [27]. Now the main idea of Laplace decomposition is to set an iteration as follows:

$$\sum_{n=0}^{\infty} e^{-n\omega s} L\left\{u_n(t)\right\} = \left(\frac{1}{s}+\frac{2}{s^2}\right)+\left(-\frac{1}{s^2}+\frac{L\left\{u_0(t)\right\}}{s}\right)e^{-\omega s} + \sum_{n=2}^{\infty}\frac{e^{-n\omega s}}{s}L\left\{u_{n-1}(t)\right\}. \qquad (2.23)$$

For $n = 0,1,2,...$, equate the co-efficient of $e^{-n\omega s}$ on both sides of (2.23) to get $L\{u_n(t)\}$.

For. $n = 0, \quad L\left\{u_0(t)\right\} = \frac{1}{s}+\frac{2}{s^2}$

For. $n = 1, \quad L\left\{u_1(t)\right\} = -\frac{1}{s^2}+\frac{1}{s}L\left\{u_0(t)\right\} = \frac{2}{s^3}$

In general, for $n \geq 2$ we have,

L . (2.24)

By using (2.22), (2.23) and (2.24), we have

$$L\left\{u(t)\right\} = \sum_{n=0}^{\infty} e^{-n\omega s} L\left\{u_n(t)\right\} = \frac{1}{s}+\frac{2}{s^2}+2\sum_{n=1}^{\infty}\frac{e^{-n\omega s}}{s^{n+2}},$$

which is same as (2.19). Therefore, by applying inverse Laplace transform, we get

$$u(t) = 1+2t+2\sum_{n=1}^{\infty}\frac{(t-n\omega)^{n+1}}{(n+1)!}e(t-n\omega), \qquad t>0. \qquad (2.25)$$

Therefore the exact solution $u(t)$ takes the following form in each interval which is same as (2.20):

$$u(t) = 1+2t+2\sum_{n=1}^{N}\frac{(t-n\omega)^{n+1}}{(n+1)!}, \qquad N\omega \leq t \leq (N+1)\omega, \quad N = 1,2,3,\ldots .$$

Therefore the Laplace decomposition method in this example generates the same exact solution achieved by the Laplace transform method. The differential-differential equation (2.1.11) with the initial interval condition (2.1.12) decreases the difference equation with the initial state to the corresponding differential equation, as $\omega \to 0$:

$u^{!}(t) - u(t) = 1, \qquad u(0) = 1 \qquad (2.26)$

and the exact solution (2.3.14) reduces to

$$u(t) = 1+2t+2\sum_{n=1}^{\infty}\frac{t^{n+1}}{(n+1)!} = 2e^t - 1.,$$

In fact, both the Laplace transform method and the Laplace decomposition method are suitable for solving (2.26) the differential-differential equation (2.26) which is made applicable. It is important to note that using either the initial conditions (2.13) or (2.17) we can deduce the same exact solution as as $\omega \to 0$.

3. RESULTS AND DISCUSSION

The Laplace decomposition method is also a perfect combination of both Laplace method of transform and Adomian decomposition method. The given example show that the Laplace decomposition method illustrates versatility and offers ease in the estimation of analytical solutions for both linear and nonlinear problems that are subject to constant and linear initial intervals condition.

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