

Solution of Einstein’s Field Equations for Gaseous Sphere

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ABSTRACT

In this paper we have considered the problem of a spherical mass of perfect fluid with variable density distribution inside the sphere and have obtained a solution of the Einstein’s field equations to study the gravitational field of a gaseous sphere. We have taken the law of

density distribution of the form $\rho = \rho_0 \left[1 - \left(\frac{r}{r_0} \right)^{2n} \right]$ where ρ_0 is the density at the

centre of sphere, r is distance from the centre and r_0 is radius of the body. This study may be useful to know the internal structures of gaseous bodies like stars.

1. Introduction

Tolman [19] developed a mathematical method for solving Einstein’s field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were the obtained and the properties of some of them were examined in detail. These solutions were used by Oppenheimer and Volkoff [14] in the study of massive neutron cores. Krori [10] obtained exact solutions for some dense massive spheres and pointed out their astrophysical implications. Mehra et al. [12] have obtained a general solution of the field equations for a composite sphere having a number of shells of different densities. Durgapal and Gehlot [3] have obtained exact internal solutions for dense massive stars in which the central pressure and density are infinitely large. Durgapal and Gehlot [45] have further obtained exact solutions for a massive sphere with two different density distributions. The density being minimum at the surface varies inversely as the square of the distance from the centre. The distribution has a core of constant density and radius. Static and non-static solutions of Einstein’s field equations have also been extensively discussed by Leibovitz [10, 11] for the spherical distribution. Some other workers are Buchdahal [12], Knutsen [7,8] and Zeldovich [23].

Singh and Yadav [17] have studied the static fluid sphere with the equation of state $p = \rho$ (i.e. stiff matter) Yadav and Saini [22] have obtained an exact, static spherically symmetric solution field equations for the perfect fluid with $p = \rho$ and a specific choice of U . To overcome the difficulty of infinite density at the centre, it is assumed that the distribution has a core of radius r_0 and constant density ρ_0 which is surrounded by the fluid with pressure equal to energy density.

As a matter of fact the field equations of general relativity theory for a spherically symmetric distribution of perfect fluid of constant density were solved by Schwarzschild [20] for the line element given by

$$ds^2 = -e^A dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^B dt^2$$

Schwarzschild obtained the exterior solution :

$$(1.1) \quad e^B = -2m/r, \quad e^A = (1 - 2m/r)^{-1}$$

and interior solutions

$$(1.2) \quad e^B = \frac{1}{4} \left(3 \left(1 - \frac{8\pi\rho R^2}{3} \right)^{1/2} - \left(1 - \frac{8\pi\rho r^2}{3} \right)^{1/2} \right)^2 \quad e^A = \left(1 - \frac{8\pi\rho r^2}{3} \right)^{-1}$$

in which m and ρ respectively denote the mass and density inside a fluid sphere. The solution (1.1) and (1.2) obtained were based on the following four assumptions.

- (i) the density of fluid inside the sphere is constant and is zero outside the sphere.
- (ii) The matter comprises of a perfect fluid at rest inside the sphere and there is no matter outside the sphere.
- (iii) The pressure is zero at the surface of the sphere and is finite and positive inside the sphere.
- (iv) The gravitational potentials of the exterior and interior solutions i.e. e^A and e^B are continuous at the boundary of the sphere.

Wyman [21] obtained the interior solution by taking the variable density distribution of fluid. He assumed the law of density to be:

$$\rho = ar^{N-2}$$

where a and N are constants and r denotes the distance from the centre. In his solution density is minimum at the centre and increases along the radius. There is also a discontinuity of density at the surface of the sphere.

But, as soon as one wishes to consider the gaseous spheres one must require that the pressure and density vanish together at the surface of the sphere.

Knutsen [9] has examined a method for constructing solutions for a relativistic ‘Gaseous’ sphere. Here he has meant ‘Gaseous’ that the density ρ vanishes at the outer boundary together with the pressure. He has discussed two different classes of solutions in detail. The models of both these classes have the property that the density gradient is zero both at the centre and at the surface. He has further shown that both the classes yield models which are physically acceptable. Some other workers in this line are Mehra [13], Pant and Sah [15].

In this paper we have considered the problem of a spherical mass of a perfect fluid with variable density distribution inside the sphere and thus obtained a solution of the Einstein’s field equations to study the gravitational field of a gaseous sphere. Here we have taken the law of density distribution to be of the form

$$(1.3) \quad \rho = \rho_0 \left[1 - \left(\frac{r}{r_0} \right)^{2n} \right]$$

Where ρ_0 is the density at the centre of the sphere, r is the distance from the centre and r_0 is the radius of the body. Here the density is maximum at the centre and decreases along the radius. The law of density has been taken like this to ensure that the density vanishes at the surface. This study differs from Wyman’s investigation in that the density is zero at the surface in the present case. This solution, therefore, may be helpful to know the internal structures of gaseous bodies specially stars because it is believed that stars condense out of clouds of gas and dust found in the GALAXY.

2. The Field Equations and their Solutions

In the case for static system having spherical symmetry and consisting of perfect fluid, we take the metric in the form given by

$$(2.1) \quad ds^2 = -e^A dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 + e^B dt^2$$

where A and B are the functions of r alone. Since the matter comprises of a perfect fluid at rest, the components of energy – momentum tensor is given by

$$(2.2) \quad T_{ij} = (\rho + p)u_i u_j - p g_{ij}$$

where u_i is the material four-velocity with

$$(2.3) \quad g_{ij} u^i u_j = 1$$

Since we are dealing with static problem, however we can evidently write for the metric (2.1), the components of fluid velocity

$$(2.4) \quad u^1 = u^2 = u^3 = 0, u^4 = (g_{44})^{-1/2} = e^{-B/2}$$

Then

$$u_1 = u_2 = u_3 = 0, u_4 = (g_{44})^{1/2} = e^{B/2}$$

The non zero components of T_j^i are

$$(2.5) \quad T_1^1 = T_2^2 = T_3^3 = -p; T_4^4 = \rho, T_j^i = 0, i \neq j$$

where p and ρ are proper pressure and density of the fluid respectively. The non-zero components of Ricci tensor R_{ij} are

$$(2.6) \quad \left\{ \begin{aligned} R_{11} &= \frac{B''}{2} - \frac{A'B'}{4} + \frac{B'^2}{4} - \frac{A'}{r} \\ R_{22} &= e^{-A} \left\{ 1 - \frac{r}{2} (A' - B') \right\} - \\ R_{33} &= R_{22} \sin^2 \theta \\ R_{44} &= e^{B-A} \left\{ -\frac{B''}{2} + \frac{A'B'}{4} - \frac{B'^2}{4} - \frac{B'}{r} \right\} \end{aligned} \right.$$

Now Einstein's field equations in general relativity are

$$(2.7) \quad R_{ij} - \frac{1}{2} R g_{ij} = -8\pi T_{ij}$$

where R_{ij} and T_{ij} have been already defined and R is scalar of curvature tensor given by $R = R_{ij} g^{ij}$

Now using R_{ij} and T_{ij} equation (2.7). We have the system of equations for the metric (2.1)

$$(2.8) \quad 8\pi p = e^A \left(\frac{B'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2},$$

$$(2.9) \quad 8\pi \rho = e^{-A} \frac{B''}{2} - \frac{A'B'}{4} + \frac{B'^2}{4} + \frac{B' - A'}{2r}$$

$$(2.10) \quad 8\pi \rho = e^{-A} \left(\frac{A'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2}$$

In the above equations prime denotes differentiation with respect to r .

The boundary conditions to determine the values of constants of integration are :

- (i) The gravitational potentials of the exterior and interior solutions are continuous at the boundary of the sphere.
- (ii) The pressure is zero at the surface of the sphere and is finite and positive every where inside the sphere.

The exterior solution of the field equations (2.8-2.10) is Schwarzschild solution [16] which is given by (1.1)

We now proceed to obtain the interior solution of the field equations assuming the law of density:

$$\text{i.e. } \rho = \rho_0 \left[1 - \left(r^2 / r_0^2 \right)^n \right]$$

Using the above form of ρ (i.e. equation (1.3)) in equation (2.10) we get

$$(2.11) \quad 8\pi \rho_0 \left[1 - \left(\frac{r^2}{r_0^2} \right)^n \right] = \frac{e^{-A} A'}{r} - \frac{e^{-A}}{r^2} + \frac{1}{r^2}$$

Integrating (2.11), we have

$$8\pi \rho_0 \int \left[r^2 - \frac{r^{2n+2}}{r_0^{2n}} \right] dr = \int r e^{-\lambda} \lambda' dr - \int e^{-\lambda} dr + \int dr + c$$

which may be reduced to

$$(2.12) \quad e^{-A} = 1 - \frac{8\pi \rho_0}{3(2n+3)} \left[(2n+3)r^2 - \frac{3r^{2n+2}}{r_0^{2n}} \right] + \frac{c}{r}$$

where c is a constant of integration. In order to avoid a singularity at the origin c should be zero. Therefore, taking $c = 0$, equation (2.12) reduces to

$$(2.13) \quad e^{-A} = 1 - \frac{8\pi \rho_0}{3(2n+3)} \left[(2n+3)r^2 - \frac{3r^{2n+2}}{r_0^{2n}} \right]$$

From equation (2.8) and (2.9), the equation to determine B can be established as follows :

Equating the two expressions for pressure we have

$$(2.14) \quad e^{-\lambda} \left(\frac{B''}{2} - \frac{A'B'}{4} + \frac{B'^2}{4} + \frac{B'}{2r} - \frac{B'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 0$$

gives equation (2.14) may be also written as

$$(2.15) \quad 2r^3 \frac{d}{dr} \left(\frac{e^{-A} - 1}{r^2} \right) + e^{-A} (2r^2 B'' + r^2 B'^2) - e^{-A} (A'B'^2 + 2rB') = 0$$

Therefore using (2.13), (2.15)

$$(2.16) \left\{ 1 - \frac{8\pi\rho_0}{3(2n+3)} \left[(2n+3)r^2 - \frac{3r^{2n+2}}{r_0^{2n}} \right] \right\} (2r^2B'' + r^2B'^2) - B'r \left(2 - \frac{16\pi\rho_0}{2n+3} \frac{nr^{2n+2}}{r_0^{2n}} \right) + \frac{32\pi\rho_0}{(2n+3)} \frac{nr^{2n+2}}{r_0^{2n}} = 0$$

Putting $X = \frac{r^2}{r_0^2}$ in (2.16) we get

$$(2.17) \left[1 - \frac{8\pi\rho_0 r_0^2}{3(2n+3)} \{ (2n+3)x - 3x \}^{n+1} \right] (2r^2B'' + r^2B'^2) - B'r \left(2 - \frac{16\pi\rho_0 r_0^2}{2n+3} nx^{n+1} \right) + \frac{32\pi\rho_0 r_0^2}{(2n+3)} nx^{n+1} = 0$$

Now substituting $\exp\left(\frac{B}{2}\right) = \psi$ in (2.17) we see

$$B' = \frac{2\psi'}{\psi}, \quad 2B'' = \frac{4\psi''}{\psi}$$

Then (3.2.17) transforms into

$$(2.18) \quad 4 \left[1 - \frac{8\pi\rho_0 r_0^2}{3(2n+3)} \{ (2n+3)x - 3x^{n+1} \} \right] r^2 \psi'' - 2 \left(2 - \frac{16\pi\rho_0 r_0^2}{2n+3} nx^{n+1} \right) \psi \psi' + \frac{32\pi\rho_0}{(2n+3)} nx^{n+1} \psi = 0$$

But $X = \frac{r^2}{r_0^2} \Rightarrow \frac{dx}{dr} = \frac{2r}{r_0^2}$ and $\frac{d^2x}{dr^2} = \frac{2}{r_0^2}$, etc. using the above results in (2.18) we get

$$(2.19) \quad \left[\frac{2n+3}{8\pi r_0^2 \rho_0} - \frac{2n+3}{3} x + x^{n+1} \right] \frac{d^2\psi^2}{dx^2} + \frac{1}{6} [(3n+3)x - (2n+3)] \cdot \frac{d\psi}{dx} + \frac{nx^{n+1}}{4} \psi = 0$$

Changing (2.19) in the general form of linear differential equation of second order

$$\frac{d^2y}{dx^2} + P \frac{dy}{dx} + Qy = R$$

where P, Q, R are functions of x, we have

$$(2.20) \quad \frac{d^2\psi}{dx^2} + \left[\frac{\left(\frac{n+1}{2} x - \frac{2n+3}{6} \right)}{\frac{2n+3}{8\pi r_0^2 \rho_0} - \left(\frac{2n+3}{3} \right) x + x^{n+1}} \right] \frac{d\psi}{dx}$$

$$+ \frac{nx^{n-1}}{4 \left[\frac{2n+3}{8\pi r_0^2 \rho_0} - \left(\frac{2n+3}{3} \right) x + x^{n+1} \right]} \psi = 0$$

(2.20) can be solved to get $\psi = e^{B/2}$. However to avoid mathematical complexity, we take $n = 1$, so that we have

$$(2.21) \quad \frac{d^2\psi}{dx^2} + \frac{x - 5/6}{\left[x^2 - \frac{5}{3}x + \frac{5}{8\pi r_0^2 \rho_0} \right]} \frac{d\psi}{dx} + \frac{1}{4 \left[x^2 - \frac{5}{3}x + \frac{5}{8\pi r_0^2 \rho_0} \right]} \psi = 0$$

Now we change the independent variable x to μ in the following way (Rainville and Bendit [18])

$$\text{Let } \left(\frac{d\mu}{dx} \right)^2 = \frac{1}{4 \left(x^2 - \frac{5}{3}x + \frac{5}{8\pi r_0^2 \rho_0} \right)}$$

$$\text{or } \frac{d\mu}{dx} = \frac{1}{2} \left(x^2 - \frac{5}{3}x + \frac{5}{8\pi r_0^2 \rho_0} \right)^{-1/2}$$

Integrating it w.r.t. x , we obtain

$$\mu = \int \frac{1}{2 \left[\left(x - \frac{5}{6} \right)^2 + \left(\frac{5}{8\pi r_0^2 \rho_0} - \frac{5}{6} \right)^2 \right]^{1/2}} dx + c$$

Which gives (taking constant of integration $c = 0$)

$$(2.22) \quad \mu = \log \left[x - \frac{5}{6} + \left(x^2 - \frac{5x}{3} + \frac{5}{8\pi r_0^2 \rho_0} \right)^{1/2} \right]$$

Now with the relationship (2.22) between z and x , we transform (2.19) with $n = 1$, to get an equation of the form

$$\frac{d^2\psi}{d\mu^2} + P_1 \frac{d\psi}{d\mu} + Q_1 \psi = R_1$$

$$\text{where } P_1 = \frac{\frac{d^2\mu}{dx^2} + \frac{pd\mu}{dx}}{\left(\frac{d\mu}{dx} \right)^2}, Q_1 = \frac{Q}{\left(\frac{d\mu}{dx} \right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{d\mu}{dx} \right)^2}$$

We see that

$$P_1 = 0, Q_1 = 1, R_1 = 0$$

So that (2.21) goes to the form

$$(2.23) \quad \frac{d^2\psi}{d\mu^2} + \frac{\psi}{4} = 0$$

The solution of (2.23) is

$$(2.24) \quad \psi = \lambda \cos(\mu/2) + \delta \sin(\mu/2)$$

Where λ and δ are integration constants. Thus we get the value of $e^B = \psi^2$ as

$$(2.25) \quad e^B = \left[\lambda \cos(\mu/2) + \delta \sin\left(\frac{\mu}{2}\right) \right]^2$$

From equation (2.8), (2.13) and (2.25) we see that the pressure at any point is given by

$$(2.26) \quad p = \left(\frac{\rho_0}{10\pi r_0^2} \right)^{1/2} \left[1 - \frac{8\pi r_0^2 \rho_0}{15} (5x - 3x^2) \right]^{1/2} \times \left(\frac{\delta - \lambda \tan \mu / 2}{\lambda + \delta \tan \mu / 2} \right) + \frac{\rho_0}{15} (3x - 5)$$

Equation (1.1) and (2.13) with the continuity of e^A (for $n = 1$) gives

$$(2.27) \quad m = \frac{8\pi r_0^3 \rho_0}{15}$$

Equation (1.1) and (2.25) with the continuity of e^B provides

$$(2.28) \quad \lambda \cos(\mu / 2) + \delta \sin(\mu / 2) \left(1 - \frac{2m}{r_0} \right)^{1/2}$$

The condition that pressure should be zero at the surface (boundary) of the sphere provide,

$$(2.29) \quad \left(\frac{8\pi r_0^2 \rho_0}{5} \right)^{1/2} \left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} \left(\frac{\delta - \lambda \tan \mu_1 / 2}{\lambda + \delta \tan \mu_1 / 2} \right) = \frac{8\pi r_0^2 \rho_0}{15}$$

where

$$(2.30) \quad \mu_1 = \log \left[\frac{1}{6} + \left(\frac{5}{8\pi r_0^2 \rho_0} - \frac{2}{3} \right)^{1/2} \right]$$

Equation (2.28) and (2.29) with equation (2.27) give

$$(2.31) \quad \lambda = \left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} \cos \frac{\mu_1}{2} - \frac{2r_0}{3} \left(\frac{2\pi \rho_0}{5} \right)^{1/2} \sin \frac{\mu_1}{2}$$

$$(2.32) \quad \delta = \left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} \sin \frac{\mu_1}{2} + \frac{2r_0}{3} \left(\frac{2\pi \rho_0}{5} \right)^{1/2} \cos \frac{\mu_1}{2}$$

The complete interior solution of the line element (2.1) is, therefore :

$$(2.33) \quad e^{-A} = 1 - \frac{8\pi \rho_0}{15} \left(5r^2 - \frac{3r^4}{r_0^2} \right),$$

$$(2.34) \quad e^B = \left[\left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} \cos \frac{\mu_1 - \mu}{2} - \frac{2r_0}{3} \left(\frac{2\pi \rho_0}{5} \right)^{1/2} \times \sin \frac{\mu_1 - \mu}{2} \right]^2$$

$$(2.35) \quad p = \frac{1}{2\pi r_0^2} \left[\left(\frac{2\pi r_0^2 \rho_0}{5} \right)^{1/2} \left\{ 1 - \frac{8\pi r_0^2 \rho_0}{15} (5x - 3x^2) \right\}^{1/2} \times \frac{\left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} \tan \frac{\mu_1 - \mu}{2} + \frac{2}{3} \left(\frac{2\pi r_0^2 \rho_0}{5} \right)^{1/2} - \frac{2\pi r_0^2 \rho_0}{15} (5 - 3x)}{\left(1 - \frac{16\pi r_0^2 \rho_0}{15} \right)^{1/2} - \frac{2}{3} \left(\frac{2\pi r_0^2 \rho_0}{5} \right)^{1/2} \tan \frac{\mu_1 - \mu}{2}} \right]$$

where \square and \square_1 are given by equations (2.22) and (2.30) respectively.

3. Remarks and Discussion

To get a real solution from equations (2.27) and (2.30) we must have:

$$(3.1) \quad r_0^2 < \frac{9}{10} \pi \rho_0, \quad m < \frac{12r_0}{25}$$

Which give an upper limit of the possible size from given density and of the mass for given radius. Again the condition (3.1) corresponds to the Tolman's condition

$$(3.2) \quad r_0^2 < \frac{3}{8} \pi / \rho_0, \quad m < \frac{r_0}{2}$$

for Schwarzschild interior solution i.e. solution III.

Further in this paper we have discussed the static field equation of general relativity theory for a particular law of density distribution. The assumed law gives us the maximum density at the centre and the density decreases along the radius. It goes to zero continuously at the surface of the sphere. Therefore this solution helps us to determine the internal structure of gaseous bodies specially stars.

If we put $n = 1$ in equation (1.3), we get the results due to Mehra [13]. Hence our work can be considered as generalization of that due to Mehra [13]

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