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Differential Equation and Mathematical Model of Population

[1]Dr. Prabir Kumar Chakraborty, [2]Dr.Vijay Kumar and [3]Irsahd Ali

[1]*Department of Mathematics, M.J.K. College Bettiah , Bihar, India.*
[2]*PG Department of Mathematics, College of Commerce, Arts and Science, Patna-20, Bihar, India.*
[3]*Department of BCA,M.J.K. College Bettiah , Bihar, India.*

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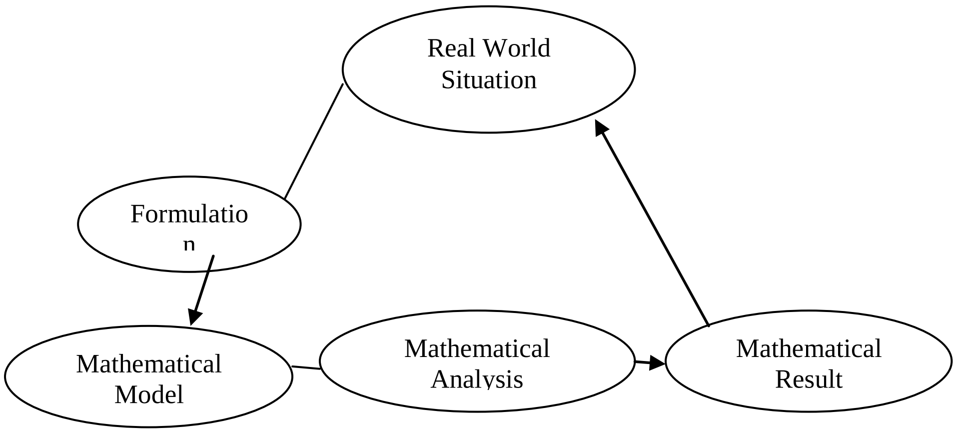
ABSTRACT

We discuss the formulation of a real world problem of population in mathematical terms; i.e. the construction of a mathematical model.
The analysis of the resulting mathematical problem.
The interpretation of the mathematical result in the context of the original real world situation.



1. Introduction

In the population example, the real world problem is that of determining the population at some future time. A mathematical model consists of a list of variables (p and t) that described the given situation together with one or more equations relating these variables.

$$\frac{dp}{dt} = kp(t), \quad p(0) = p_0$$

The mathematical analysis consists of solving these equations. Finally, we apply these mathematical results to attempt to answer the original real-world questions.

Nevertheless, it is quite possible that no one solution of the differential equation fits all the know in-formation. In such a case we must suspect that the differential equation may not adequately the rate of change of a population p(t) with constant birth and death rates is proportions' to the size of population i.e.

$\frac{dp}{dt} \propto p \Longrightarrow \frac{dp}{dt} = kp$, where k is constant.

$$\ln p = kt + \ln c \quad \Longrightarrow \frac{p}{c} = e^{kt}$$

$p(t) = ce^{kt}$ is a solution of the differential equation.

$p(t)$accurately described the actual growth of the human population of the world over of the past few countries. We must therefore write a perhaps more complicated differential equation, one that takes into account the effects of population pressure as the birth rate, the decline food supply and other factors.

In our population example we ignored the effects of such factores as varying birth and death rates. This made the mathematical analysis quite simple, perhaps, unrealistically so. A satisfactory mathematical model is subject to two contradictory requirements. It must be sufficiently detailed to represent the real world situation, then the mathematical analysis may be too difficult do carry out. If the model is too simple, the result may be so in accurate as to be useless. The construction of a model that adequately bridges this gap between realism and feasibility is therefore the most crucial and delicate step in the process.

2. Natural growth and decay:-

The differential equation $\frac{dp}{dt} = kx$, (k as a constant) serves as a mathematical model for a remarkably wide range of natural phenomena and involving a quantity whose time rate of change proportional to its current size.

3. population growth:-

Suppose that $p(t)$ is the number individuals in population of human having constant birth and death rates β and δ (in birth or death per individual per unit of time). Then, during a short time interval Δt, approximately β $p(t)$ Δtdeaths occur, so the change in $p(t)$ is:

$$\Delta p = (\beta - \delta)p(t)\Delta t$$

$\frac{dp}{dt} = \lim_{\Delta \to 0} \frac{\Delta p}{\Delta t} = kp$ where $k = \beta - \delta$

4. Population model:

As we have discussed, the exponential differential equation $\frac{dp}{dt} = kp$ with solution$p(t) = p_0 e^{kt}$, as a mathematical model for natural growth of population that occur as a result of constant birth rates. Here we present a more general population model that accommodates birth and death rates that are not necessarily constant. However, our population function $p(t)$will be a continuous approximation to the actual population, which of course grows by integral increments.

Suppose that the population changes only by the occurrenece of birth and death. There is no immigration or emigration from outside the country or environment under consideration. It is customary to track the growth or decline of a population in terms of its birth rate and death rate function.

$\beta(t)$: The number of births per unit of population per unit of time at t

$\delta(t)$: The number of deaths per unit of population per unit of time at t

The number of births and deaths that occurrence during the time interval $[t, t + \Delta t]$ is births : $\beta(t)p(t)\Delta t$, deaths : $\delta(t)p(t)\Delta t$.

Hence the change ΔP in the population during the time interval $[t, t + \Delta t]$ is :

Δp = birth – death = : $[\beta(t)p(t) - \delta(t)p(t)]\Delta t$

$$\therefore \frac{\Delta P}{\Delta t} = (\beta(t) - \delta(t))p(t)$$

The error in this approximation should approach zero as $\Delta t \to 0$, so

$\frac{dp}{dt} = (\beta - \delta)p$ .... (i)

Hence equation (i) is the general population eqution. If β and δ are constant then equation (i) reduces to natural growth equation with$k = \beta - \delta$. But it also include the possibility that β and δ are variable function of t, so the birth and death rates depend on $p(t)$.

5. Bounded population and logistic equation:

In situations as diverse as the human population of a nation and a fruit fly population in a closed container, it is often observed that the birth rate decrease as the population itself increases.

Suppose that the birth rate β is a linear decreasing function of population size p, so that $\beta = \beta_0 - \beta_1 p$, where β_0 and β_1 are positive constant. If the death rate $\delta = \delta_0$, remains constant, then from (i) :

$\frac{dP}{dt} = (\beta_0 - \beta_1 p - \delta_0)p$, i.e. $\frac{dP}{dt} = ap - bp^2$ .... (ii)

Where $a = \beta_0 - \delta_0,\ b = \beta_1$.

Where a and b are constant positive. Then the equation (ii) is called logistic equation

For the purpose of relating the behaviour of the population $p(t)$ to the values of the parameters in equation. It is useful to rewrite the logistic equation $\frac{dP}{dt} = kp(m - p)$, where k=b and $m = \frac{a}{b}$

Limiting population and carrying capacity, the finite population is a

Characteristic of logistic population

$$\frac{dP}{dt} = kp(m - p), p(0) = p_0$$

$p(t) = \frac{mp_0}{p_0 + (m - p_0)\, e^{-kMt}}$, ... (iii)

Actual animal population are positive value. If $p_0 = m$, then (iii) reduces to the unchanging (constant-valued), equilibrium population

$p(t) = m$. Otherwise, the behaviour of a logistic population depends on $0 < p_0 < m$ or $p_0 > m$, if $0 < p_0 < m$:

$p(t) = \frac{mp_0}{p_0 + (m - p_0)\, e^{-kMt}} = \frac{mp_0}{p_0 + (\text{positive no.})} < \frac{mp_0}{p_0} = m$, ... (iv)

However if $p_0 > m$, then

$p(t) = \frac{mp_0}{p_0 + (m - p_0)\, e^{-kmt}} = \frac{mp_0}{P_0 + (\text{negative no.})} > \frac{mp_0}{p_0} = m$ ...... (v)

$\lim_{t\to\infty} p(t) = \frac{mp_0}{p_0 + 0} = m.$ .... (vi)

Thus a population that satisfies the logistic equation does not grow without bound like a naturally growing population.

The population $p(t)$ steadily increases and approaches m from below $0 < p_0 < m$, but steadily decreases and approaches m from above if $p_0 > m$. Sometimes m is called carrying capacity of the environment.

6. Historical note:

The logistic equation was introduced in 1840 by the Belgium mathematician and demographer P.F. Verhulstas a possible model for the human population growth.

7. Application of the logistic equation:

(1) **Limited environment situation**:- A certain environment can support a population of at most m individual. It is then reasonable to expect the growth rate proportional to $m - p$.

i.e.$\frac{dP}{dt} = kp(m - p)$

(2) Competition situation if the birth rate β is constant but the death rate $\delta$is proportional to p

$$\frac{dP}{dt} = (\beta - \delta)p = kp(m - p).$$

This might be a reasonable working hypothesis in a study of a Cannibalistic population.

(3) These investigation deal with the problem of fitting a logistic model to given population data.

$\frac{dP}{dt} = aP + bP^2, P(0) = P_0.$

8. World Population Date:

Year	World Population(billions)
1960	3.049
1970	3.721
1980	4.473
1990	5.249
2000	6.127
2015	7.349

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