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Optimization of Various Measures of Fuzzy Entropy

[1]Sajan Bedi & [2]Anika Indoria

[1,2]Department of Mathematics, S.S.M. College Dinanagar (India)

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ABSTRACT
Optimization problems play an important role in the literature of information theory. Keeping in view the importance and areas of applications of these optimization problems, we have investigated the optimum values of various measures of fuzzy entropy.

1. INTRODUCTION

In most of the real life situations, uncertainty arises in decision-making problem either due to lack of knowledge or due to inherent vagueness. Such types of problems can be solved with the help of probability theory and fuzzy set theory, respectively. In the literature of theory of fuzzy information, a measure of fuzziness which is often used and citied is certainly an entropy function first mentioned by Zadeh [12]. However, the two functions measure fundamentally different types of uncertainty. Basically, the Shannon's
[11] entropy measures the average uncertainty in bits associated with the prediction of outcomes in a random experiment. De Luca and Termini [2] introduced some requirements which capture our intuitive comprehension of the degree of fuzziness and consequently developed a measure of fuzzy entropy which corresponds to Shannon's [11] entropy. This measure is given by

$$H(A) = -\sum_{i=1}^{n}\left[\mu_A(x_i)\log\mu_A(x_i) + (1-\mu_A(x_i))\log(1-\mu_A(x_i))\right] \quad (1.1)$$

Kapur [5] took the following measure of fuzzy entropy corresponding to Havrada and Charvat's [4] probabilistic entropy:

$$H^{a}(A) = (1-a)^{-1} = -\sum_{i=1}^{n}\left[\{\mu^{\alpha}_A(x_i) + (1-\mu_A(x_i))^{\alpha}\} - 1\right];\alpha\neq 1,\alpha>0 \quad (1.2)$$

Bhandari and Pal [1] developed the following measure of fuzzy entropy corresponding to Renyi's [10] probabilistic entropy:

$$H_{\alpha}(A) = \frac{1}{1-\alpha}\sum_{i=1}^{n}\log\left[\mu_A^{\alpha}(x_i) + (1-\mu_A(x_i))^{\alpha}\right];\alpha\neq 1,\alpha>0 \quad (1.3)$$

Parkash [7] introduced a new generalized measure of fuzzy entropy involving two real parameters, given by

$$H_{\alpha}(A) = [(1-\alpha)\beta]^{-1}\sum_{i=1}^{n}\left[\left\{\mu_A^{a}(x_i) + (1-\mu_A(x_i))^{a}\right\}^{\beta} - 1\right];\alpha>0,\alpha\neq 1,\beta\neq 0 \quad (1.4)$$

and called it ($\alpha-\beta$) fuzzy entropy which includes some well known fuzzy entropies. Some other interesting findings related with theoretical measures of fuzzy entropy and their applications have been investigated by Pal and Bezdek [8], Ebanks [3], Parkash, Sharma and Mahajan [8], Parkash and Tuli [9] etc.

2. OPTIMUM VALUES OF VARIOUS FUZZY ENTROPIES

In this section, we consider Renyi's [10] measure of fuzzy entropy and examine it for its maximum and minimum values.

I. Minimum Values of Renyi's [10] measure of fuzzy entropy

Our purpose is to find the minimum value of $H_{\alpha}(A)$ given in (1.3) subject to the following constraint:

$$\sum_{i=1}^{n} \mu_A(x_i) = k,\ 0 \le k \le n. \tag{2.1}$$

For this purpose, we consider the following cases:

Case-I: When k is any positive integer, then we can choose k values of $\mu_A(x_i)$ as unity and other $(n-k)$ values as 0, that is, $\mu_A(x_i) = \{1,1,...,1,0,0,...,0\}$

Thus, we rewrite equation (1.3) as follows:

$$H_\alpha(A) = \frac{1}{1-\alpha}\left[\sum_{i=1}^{m} \log\left(u^\alpha{}_A(x_i) + (1-\mu_A(x_i))^\alpha\right) + \sum_{i=m+1}^{n}\left(u^\alpha{}_A(x_i) + (1-\mu_A(x_i))^\alpha\right)\right]$$

The minimum value of $H_\alpha(A)$ is given as

$$Min.H_\alpha(A) = \frac{1}{1-\alpha}\left[n\log 1 + n\log 1\right] = 0$$

Case-II: If k is any fraction, then, we can write $k = m + \xi$, where m is a positive integer and ξ is a positive fraction. We can thus choose m fuzzy values of $\mu(x)$ as unity, $(m+1)^{th}$ value of $\mu_A(x_i)$ as ξ and remaining $(n-m-1)$ fuzzy values$^{A\,i}$ of $\mu_A(x_i)$ as 0, that is, $\mu_A(x_i) = \{1,1,1,.....,1,\xi,0,0,0,..., 0\}$. Now,

$$H_\alpha(A) = \frac{1}{1-\alpha}\left[\begin{array}{l}\sum_{i=1}^{m} \log\left(\mu^\alpha{}_A(x_i) + (1-\mu_A(x_i))^a\right) + \log\left(\mu^\alpha{}_A(x_{m+1}) + (1-\mu_A(x_{m+1}))^a\right) \\ + \sum_{i=m+1}^{n} \log\left(\mu^\alpha{}_A(x_i) + (1-\mu_A(x_i))^\alpha\right)\end{array}\right]$$

Thus, $Min.H_\alpha(A) = \dfrac{1}{1-\alpha}\log\left[\xi^\alpha + (1-\xi)^\alpha\right]$

Taking $\phi(\xi) = \xi^\alpha + (1-\xi)^\alpha,\ 0 \le \xi \le 1$, then

$$\phi'(\xi) = \alpha\left[\xi^{\alpha-1} - (1-\xi)^{\alpha-1}\right]$$

$$\phi''(\xi) = \alpha(\alpha-1)\left[\xi^{\alpha-2} + (1-\xi)^{\alpha-2}\right] = \left(\alpha^2 - \alpha\right)\left[\xi^{\alpha-2} + (1-\xi)^{\alpha-2}\right]$$

Case- I. When $\alpha > 1$, then $\phi''(\xi) > 0$. Thus, $\phi(\xi)$ is a convex function of ξ

$\Rightarrow \log\phi(\xi)$ is also a convex function of ξ

$\Rightarrow \dfrac{1}{1-\alpha}\log\phi(\xi)$ is a concave function of ξ for $\alpha > 1$

Case-II. When $\alpha < 1$, then $\phi''(\xi) < 0$

$\Rightarrow \phi(\xi)$ is a concave function of ξ

$\Rightarrow \dfrac{1}{1-\alpha}\log\phi(\xi)$ is a concave function of ξ for $\alpha > 1$

Hence, $Min.H_{\alpha,\beta}(A)$ is a concave function of ξ for each value of α. Hence, its maximum value exists.
For maximum value of $Min.H_\alpha(A)$, we put

$$\frac{d\left(Min.H_\alpha(A)\right)}{d\xi} = 0 \text{, which gives } \xi = 1/2$$

Thus, the maximum value of $Min.H_\alpha(A)$ exists at $\xi = \frac{1}{2}$. Also, the maximum value of $Min.H_\alpha(A)$ is given as

$$Max.Min.H_\alpha(A) = \frac{1}{1-\alpha}\log\left[\left(\frac{1}{2}\right)^\alpha + \left(\frac{1}{2}\right)^\alpha = \right] \frac{1}{1-\alpha} \log 2^{1-\alpha} = \log 2$$

Again when $\xi = 0,\ Min.H_{\alpha,\beta}(A) = 0$

When $\xi = \frac{1}{2},\ Min.H_{\alpha,\beta}(A) = \log 2$

When $\xi = 1,\ Min.H_{\alpha,\beta}(A) = 0$

Thus, $Min.H_\alpha(A)$ increases from 0 to $\log 2$ as ξ increases from 0 to $\frac{1}{2}$ and decreases from $\log 2$ to 0 as ξ further increases from $\frac{1}{2}$ to 0 as shown below in Fig.-2.1:

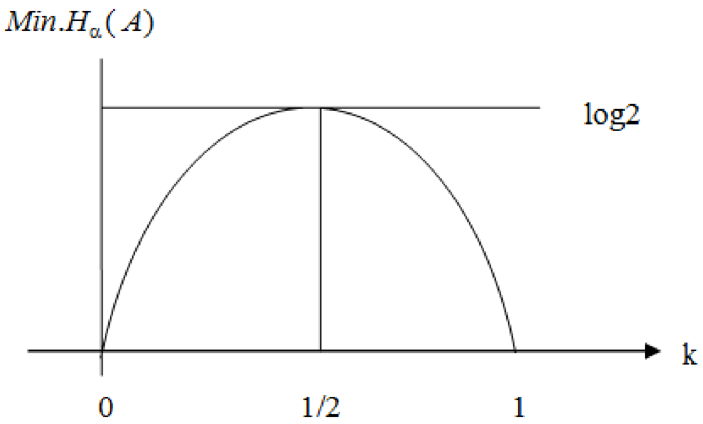

Fig-2.1

II. Maximum Values of Renyi's [10] measure of fuzzy entropy:

Here, our purpose is to find maximum values of $H_\alpha(A)$ subject to the constraints (2.1).

Taking $f(x) = x^\alpha + (1-x)^\alpha$, $0 \le x \le 1$, we have

$f''(x) = \alpha(\alpha-1)\left[x^{\alpha-2} + (1-x)^{\alpha-2}\right] = (\alpha^2 - \alpha)\left[x^{\alpha-2} + (1-x)^{\alpha-2}\right]$

Case-I. When $\alpha > 1$, then $f''(x) > 0$

Then $f(x)$ is a convex function of x where $0 \le x \le 1$.

$\Rightarrow \frac{1}{1-\alpha}\log f(x)$ is also a concave function of x

$\Rightarrow H_\alpha(A)$ is also a concave function of $\mu_A(x_i)$ for $\alpha > 1$

Case-II. When $\alpha < 1$, then $f''(x) < 0$

Then $f(x)$ is a concave function of x where $0 \le x \le 1$.

$\Rightarrow H_\alpha(A)$ is also a concave function of $\mu_A(x_i)$ for $\alpha < 1$

Hence, we have proved that $H_\alpha(A)$ is a concave function for each value of α and thus its maximum value exists at $\mu_A(x_i) = \frac{1}{2}$.

Now $\sum_{i=1}^{n} \mu_A(x_i) = k$ gives $\frac{n}{2} = k \Rightarrow \frac{k}{n} = \frac{1}{2} = \mu_A(x_i) \forall i$

Also, the maximum value of $H_\alpha(A)$ is given as

$$\text{Max.}H_\alpha(A) = \frac{1}{1-\alpha} \sum_{i=1}^{n} \log \left[\left(\frac{k}{n}\right)^\alpha + \left(1 - \frac{k}{n}\right)^\alpha \right] = \frac{n}{1-\alpha} \sum_{i=1}^{n} \log \left[\frac{k^\alpha + (n-k)^\alpha}{n^\alpha} \right]$$

Take $g(k) \frac{k^\alpha + (n-k)^\alpha}{n^\alpha}$; $g''(k) = (\alpha^2 - \alpha) \left[\frac{k^\alpha + (n-k)^\alpha}{n^\alpha} \right]$

Case-I. When $\alpha > 1$, then $g''(k) > 0$

$\Rightarrow g(k)$ is a convex function of k

$\Rightarrow \frac{n}{1-\alpha} \log g(k)$ is a concave function of k for $\alpha > 1$

Case-II. When $\alpha < 1$, then $g'(k) < 0$

$\Rightarrow \frac{n}{1-\alpha} \log g(k)$ is a concave function of k for $\alpha < 1$

Hence, $Max.H_\alpha(A)$ is a concave function of k. Hence, the maximum value of $Max.H_\alpha(A)$ exists at $\frac{k}{n} = \frac{1}{2} = \mu_A(x_i) \forall i$.

Also, $Max.Max.H_\alpha(A) = \frac{n}{1-\alpha} \log \left[\left(\frac{1}{2}\right)^\alpha + \left(\frac{1}{2}\right)^\alpha \right] \frac{n}{1-\alpha} = \log 2^{1-\alpha} = n\log 2$

When $k = 0$, then $Max.H_\alpha(A) = \frac{n}{1-\alpha} \log 1 = 0$

ALSO, $\frac{d(\text{Max.}H_\alpha)(A)}{dk} = \frac{n}{1-\alpha} \left(\frac{-\alpha n^{\alpha-1}}{n^\alpha} \right) \frac{\alpha}{\alpha - 1} \begin{cases} >0, \alpha>1 \\ <0, \alpha<1 \end{cases}$

When $k = \frac{n}{2}$, then $Max.H_\alpha(A) = \frac{n}{1-\alpha} \log \left[\left(\frac{1}{2}\right)^\alpha + \left(\frac{1}{2}\right)^\alpha \right] = \frac{n}{1-\alpha} \log 2^{1-\alpha} = n \log 2$ and

Also, $\frac{d(Max.H_\alpha(A))}{dk} = 0$

When $k = n$, then $Max.H_\alpha(A) = 0$ and

$$\frac{d(Max.H_\alpha(A))}{dk} = \frac{n}{1-\alpha}\left(\frac{\alpha n^{\alpha-1}}{n^\alpha}\right) = \frac{\alpha}{\alpha-1}\begin{cases} >0, \alpha<1 \\ <0, \alpha>1 \end{cases}$$

At $k = 0$, the graph of $Max.H_\alpha(A)$ is monotonically increasing for $\alpha > 1$ and monotonically decreasing if $\alpha < 1$

At $k = \frac{n}{2}$, the tangent is parallel to k-axis.

At $k = n$, the graph of $Max.H_\alpha(A)$ is monotonically decreasing for $\alpha > 1$ and monotonically increasing if $\alpha < 1$. Thus, $Max.H_\alpha$,β (A) increases from 0 to $n \log 2$ as k increases from 0 to $\frac{n}{2}$ and decreases from $n \log 2$ to 0 as k further increases from $\frac{n}{2}$ to n as shown below in Fig.-2.2.

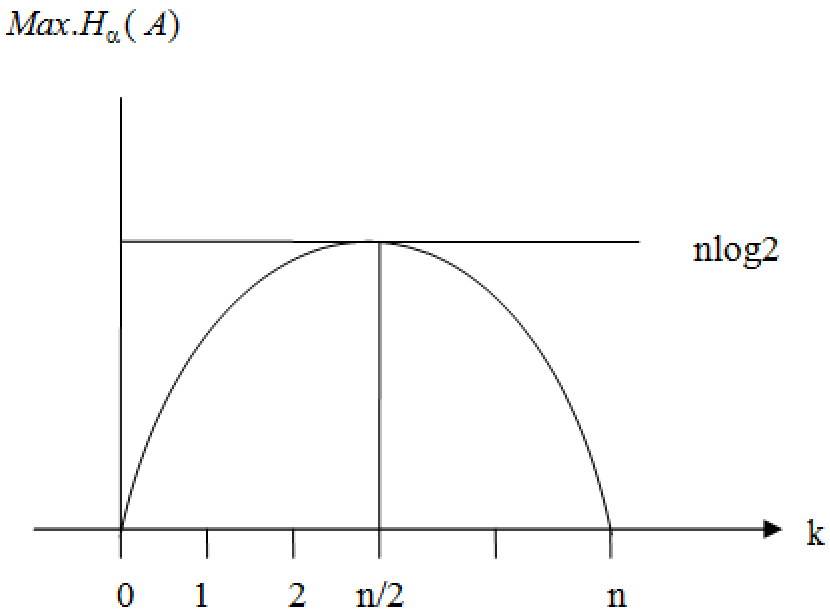

Fig-2.2

3. CONCLUDING REMARKS:

There exist many parametric and non-parametric measures of fuzzy entropy introduced by various researchers. Proceeding on similar way as done in section 2, the optimum values of other measures can be studied.

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