RESEARCH REVIEW JOURNALS
Volume-05
Issue-03
March -2020
ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[Peer Reviewed Journal]

By Solutions of the Equation Δu=f, Find Uniform Approximation on Riemannian Manifolds

Dr. Shashi Bhushan Rai

Associate Professor, Department of Mathematics, B.N.College Patna

ARTICLE DETAILS

Article History
Published Online: 16 Mar 2020

Keywords
Harmonic functions, Uniform approximation, Partial differential equations, Harmonic approximation, Riemannian manifolds.

ABSTRACT

The purpose of this paper is to present several results of Runge type on uniform approximation by solutions of the equation$\Delta u=f$ onC^{∞}Riemannian manifolds.

1. Introduction

Let Ω be an orientable Riemannian manifold and let Δ be the Laplacian of Ω.In this paper, we prove several approximation theorems for solutions of the equation$\Delta \mathrm{u} = \mathrm{f}$ where $\mathrm{f} \in \mathrm{C}^{\infty}(\Omega)$.Since f is C^{∞},any solution u of $\Delta u = f$ is necessarily $C^{\infty}$by elliptic regularity theory.

We say that a continuous function$u: \Omega \to [\infty \ - \infty]$ is **Newtonian**

provided that there is a discrete set$A \subset \Omega$with the following properties:

(i) $_{A}\, u\, _{\backslash\Omega}$is harmonic, and

(ii) for each point a $\in$A,there is a regular subregion R satisfying $R \cap A=\{a\}$and a constant c, such that the function$u - c\, G_R(\bullet, a)$ can be defined at a so as to be harmonic on R, where$G_R(\bullet, a)$ is the Green function of R.

We say that a continuous function$u: \Omega \to [\infty \ - \infty]$is a **singularsolution** of the equation$\Delta \mathbf{u} = \mathbf{f}$ provided$u = v + w$where v is Newtonian on Ω and where w is a regular(C^{∞})solution of $\Delta \mathbf{w} = \mathbf{f}$on Ω.

Thus$\Delta \mathbf{u} = \mathbf{f}$on Ω except possibly for isolated singularities.

Lemma 1. *Let* Ω *be an orientable compact* C^{∞}*Riemannian manifold and let*$\mathrm{f} \in \mathrm{C}^{\infty}(\Omega)$. *There exists* $\mathrm{u} \in C^{\infty}(\Omega$*such that* $\Delta \mathbf{u} = \mathbf{f}$ on Ω.*if and only if* $\int \mathrm{f} d\lambda = 0$.*The solution u is unique up to a constant.*

Theorem 1. *Let* Ω *be an orientable compact*C^{∞}*Riemannian manifold and let*$\mathrm{f} \in \mathrm{C}^{\infty}(\Omega)$*such that* $\int \mathrm{f} d\lambda = 0$.*Let F be a closed subset of* Ω *and let u be a solution of* $\Delta \boldsymbol{u} = \boldsymbol{f}$*on an open set containing F.*

If $0 > \varepsilon$, *then there exists a singular solution v of* $\Delta \mathbf{v} = \mathbf{f}$*on* Ω *such that* $\sup_{\mathrm{F}} |\mathrm{u} - \mathrm{v}| < \varepsilon$

Proof. We may assume that F is not all of ,Ω and that u is a solution of $\Delta \boldsymbol{u} = \boldsymbol{f}$on an open set $\mathrm{N} \supset \mathrm{F}$,which is not dense in Ω.By Lemma 1, there exists $\mathrm{u}_1 \in \mathrm{C}^{\infty}(\Omega)$such that $\Delta \boldsymbol{u_1} = \boldsymbol{f}$on Ω.Since $\Delta(\mathrm{u} - \mathbf{u_1}) = \Delta \mathrm{u} - \Delta \mathbf{u_1} = \mathbf{f} - \mathbf{f} = \mathbf{0}$on N we see that $\mathrm{u} - \boldsymbol{u_1}$is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function h on Ω such that $\sup_{\mathrm{F}|\mathrm{u} - \mathbf{u_1} - \mathrm{h}\ | < \varepsilon}$.We set $\mathrm{v} = \mathrm{h} + \mathbf{u_1}$.Thus v is a singular solution of the equation $\Delta \boldsymbol{v} = \boldsymbol{f}$on Ω which has the required property.

Theorem-2. *Let* Ω *be an orientable compact* C^{∞}*Riemannian manifold and let* $\mathrm{f} \in \mathrm{C}^{\infty}(\Omega)$*such that* $\int \mathrm{f} d\lambda = 0$.*Let F be a closed subset of* Ω *and let u be a singular solution of* $\Delta \boldsymbol{u} = \boldsymbol{f}$*on an open set containing F. If* $\varepsilon > 0$*then there exists a singular solution v of* $\Delta \boldsymbol{v} = \boldsymbol{f}$*on* Ω *such that*$\sup_F |u - v| < \varepsilon$

Proof. We may assume that F is not all of Ω and that u is a singular solution of $\Delta \mathbf{v} = \mathbf{f}$on an open set$N \supset F$,which is not dense in Ω.

Then we may write$u = v + w$,where v is Newtonian on N and where w is a regular C^{∞}solution of $\Delta \mathbf{w} = \mathbf{f}$on N.

Therefore, there exists a Newtonian function $v_1$on Ω such that $v - v_1$is harmonic on N.

By Lemma 1, there exists$w_1 \in C^\infty(\Omega)$such that $\Delta w_1 = \mathbf{f}$on Ω.Since$\Delta(w - w_1) = f - f = 0$ on N we see that $w - w_1$is harmonic on N. By [2, Theorem 5], there exists a Newtonian function $h_1$on Ω such that

$\sup_F|v - v_1 + w - w_1 - h_1| < \varepsilon$

We set $h = h_1 + v_1 + w_1$.Since the sum of two Newtonian functions is Newtonian we see that h is a singular solution of $\Delta\mathbf{h} = \mathbf{f}$on Ω such that $sup_F|u - h| < \varepsilon$.

This completes the proof of Theorem 2.

Lemma2.*Let*Ω *be an orientable noncompact*C^∞*Riemannian manifold and let* $f \in C^\infty(\Omega)$.*There exists* $u \in C^\infty(\Omega)$*such that* $\Delta\mathbf{u} = \mathbf{f}$*on* .$\Omega$

Theorem 3. Let Ω be an orientable noncompact C^∞Riemannian manifold and let$f \in C^\infty$.Let F be a closed subset of Ω and let u be a solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set containing F.

If ε>0,then there exists a singular solution v $\Delta\mathbf{v} = \mathbf{f}$on Ω such

$$\sup_F|u - v| < \varepsilon$$

Proof. Let u be a solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set N ⊃F.By Lemma 2, there exists $u_1 \in C^\infty(\Omega)$such that$\Delta\mathbf{u_1} = \mathbf{f}$on Ω.

Since $\Delta(u - u_1) = \Delta u - \Delta u_1 = f - f = 0onN$. we see that $u - u_1$is harmonic on N. By [2, Theorem 5], there exists a Newtonian function h on Ω such that$\sup_F|u - u_1 - h| < \varepsilon$.

We set $v = h + u_1$.Thus v is a singular solution of $\Delta\mathbf{v} = \mathbf{f}$on Ω which has the required property.

Theorem 4. Let Ω be an orientable noncompact C^∞Riemannian manifold and let $f \in C^\infty(\Omega)$. Let F be a closed subset of Ω and let u be a singular solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set containing F. If ε>0then there exists a singular solution h of $\Delta\mathbf{h} = \mathbf{f}$on Ω such that $\sup_F|u - h| < \varepsilon$

Proof. Let u be a singular solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set N ⊃FWe may write$u = v + w$, where$v$ is Newtonian on N and where w is a regularC^∞solution of $\Delta\mathbf{w} = \mathbf{f}$ on N. By [2, Theorem 5.3], there exists a Newtonian function $v_1$on Ω such that $v - v_1$is harmonic on N.

By Lemma 2, there exists $w_1 \in C^\infty(\Omega)$such that$\Delta w_1 = f$ on Ω.

Since $\Delta(w - w_1) = \Delta w - \Delta w_1 = f - f = 0$on N.we see that $w - w_1$is harmonic on N. By [2, Theorem 5.1], there exists a Newtonian function $h_1$on Ω such that $\sup_F|v - v_1 + w - w_1 - h_1| < \varepsilon$.

We set $h = h_1 + v_1 + w_1$.Since the sum of two Newtonian functions is Newtonian we see that h is a singular solution of $\Delta h = f$on Ω such that $sup_F|u - h| < \varepsilon$

This completes the proof of Theorem 4.

Theorem 5. Let Ω be an orientable noncompact C^∞Riemannian manifold and let $f \in C^\infty(\Omega)$ .Let F be a closed subset of Ω such that $\Omega^*\backslash F$is connected and locally connected. If u is a solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set containing F and if ε is a positive constant, then there exists $v \in C^\infty(\Omega)$ solution of $\Delta\mathbf{v} = \mathbf{f}$on Ω such that $\sup_F|u - v| < \varepsilon$

Proof. Let u be a solution of $\Delta\mathbf{u} = \mathbf{f}$on an open set N ⊃FBy **Lemma 2**, there exists $u_1 \in C^\infty($Ωsuch that $\Delta\mathbf{u_1} = \mathbf{f}$on Ω.

Since $\Delta(u - u_1) = \Delta u - \Delta u_1 = f - f = 0onN$.We see that $u - u_1$is harmonic on N. By [2, Theorem 5.3], there exists a harmonic function h on Ω such that $sup_F|u - u_{1-}v| < \varepsilon$

We set $v = h + u_{1.}$Then $v \in C^\infty(\Omega)$and$\Delta v = \Delta h + \Delta u_1 = 0 + f = f$

on Ω.In addition$\sup_F|u - v| < \varepsilon$.

This completes the proof of Theorem 5.

The definition of singular solution of $\Delta\mathbf{u} = \mathbf{f}$should be compared with our definition of sub harmonic singular function presented in [4] and [5].

References

[1] T. Aubin, Nonlinear Analysis on Manifolds, Monge-Ampère Equations, Springer- Verlag, New York, 1982.

[2] T. Bagby and P. Blanchet, Uniform harmonic approximation on Riemannian manifolds, J. Anal. Math. 62 (1994), 47-76.

[3] R. E. Greene and H. Wu, Integrals of subharmonic functions on manifolds of nonnegative curvature, Inv. Math. 27 (1974), 265-298.

[4] P. Blanchet, A Runge theorem for subharmonic functions on Riemannian manifolds, Complex Variables and Elliptic Equations 53(9) (2008), 817-818.

[5] P. Blanchet, A Runge theorem for subharmonic functions on closed subsets of Riemannian manifolds, Journal of Mathematical Sciences: Advances and Applications 31 (2015), 37-41.