ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

A Study of Complex Numbers Combination in Disciplinary in Mathematics and Calculation

[1]Priyanka & [2]Dr. Ashwini Kumar Nagpal

[1]Research Scholar OPJS University Churu Rajasthan
[2]Professor OPJS University Churu Rajasthan

ARTICLE DETAILS

Article History
Published Online: 13 March 2019

Keywords
Interdisciplinary integration, complex numbers, geometry plane, mathematics and calculation.

ABSTRACT

Complex numbers are an obligatory content of mathematical education. However, almost without exception, in all degrees of education their application is restricted to geometrical interpretation of the complex number and solving algebraic equations. Certainly, this kind of positioning of this subject content is not favorable especially if we take into consideration the opportunities for interdisciplinary integration offered by the study of complex numbers and their application in Euclid's plane geometry.

1. Introduction

The idea for curriculum integration stems from the young generation's ambition to be presented with a complete and unique idea for the nature, society and their place in them. The traditional classification of teaching content into distinct independent subjects is initiated by the ambition to provide young generations with profound knowledge from a specific area of interest, knowledge that the individual can independently connect to a whole. Therefore, the cross and inter disciplinary correlation of teaching content is of significant importance.

Practice shows that interdisciplinary correlation in mathematics teaching is accomplished on a relatively high level. However, it seems that the opportunities for mathematics teaching integration, offered by specific content topics, are not fully utilized, especially in higher degrees of education. For instance, this is the case with the study of complex numbers, the study of which in teacher training programmes in university courses the following aims are achieved:

- Realize the need to expand the set of real numbers,
- Adopt the notions for complex numbers, conjugate complex number, modulus of a complex number and operations with complex numbers,
- Adopt the geometric interpretation of complex numbers,
- Adopt the notions complex plain and Riemann sphere,
- Solve equations in the set of complex numbers, and
- Adopt the metric and topology properties of set of complex numbers, which enables students to prepare for the study of complex analysis.

2. Constructing The Complex Numbers

One way of introducing the field C of complex numbers is via the arithmetic of 2 × 2 matrices.

A complex number is a matrix of the form

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix},$$

where x and y are real numbers

Complex numbers of the form $\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$ are scalar matrices and are called real complex numbers and are denoted by the symbol {x}.

The real complex numbers {x} and {y} are respectively called the realpart and imaginary part of the complex number $\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$.

The complex number $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is denoted by the symbol i.

We have the identities

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & -y \\ y & 0 \end{bmatrix} = \begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$$

$$= \{x\} + i\{y\},$$

$$i^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \{-1\}.$$

Complex numbers of the form i{y}, where y is a non–zero real number, are called imaginary numbers.
If two complex numbers are equal, we can equate their real and imaginary parts:

$$\{x_1\} + i\{y_1\} = \{x_2\} + i\{y_2\} \Rightarrow x_1 = x_2 \text{ and } y_1 = y_2,$$

if x1, x2, y1, y2 are real numbers. Noting that {0} + i{0} = {0}, gives the useful special case is

$$\{x\} + i\{y\} = \{0\} \Rightarrow x = 0 \text{ and } y = 0,$$

if x and y are real numbers.
The sum and product of two real complex numbers are also real complex numbers:

$$\{x\} + \{y\} = \{x + y\}, \quad \{x\}\{y\} = \{xy\}.$$

Also, as real complex numbers are scalar matrices, their arithmetic is very simple. They form a field under the operations of matrix addition and multiplication. The additive identity is {0}, the additive inverse of {x} is {−x}, the multiplicative identity is {1} and the multiplicative inverse of {x} is $\{x^{-1}\}$. Consequently

$$\{x\} - \{y\} = \{x\} + (-\{y\}) = \{x\} + \{-y\} = \{x - y\},$$

$$\frac{\{x\}}{\{y\}} = \{x\}\{y\}^{-1} = \{x\}\{y^{-1}\} = \{xy^{-1}\} = \left\{\frac{x}{y}\right\}.$$

It is customary to blur the distinction between the real complex number {x} and the real number x and write {x} as x. Thus we write the complex number {x} + i{y} simply as x + iy.

More generally, the sum of two complex numbers is a complex number:

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2);$$

and (using the fact that scalar matrices commute with all matrices under matrix multiplication and {−1}A = −A if A is a matrix), the product of two complex numbers is a complex number:

$$(x_1 + iy_1)(x_2 + iy_2) = x_1(x_2 + iy_2) + (iy_1)(x_2 + iy_2)$$
$$= x_1x_2 + x_1(iy_2) + (iy_1)x_2 + (iy_1)(iy_2)$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$
$$= (x_1x_2 + \{-1\}y_1y_2) + i(x_1y_2 + y_1x_2)$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2),$$

The set C of complex numbers forms a field under the operations of matrix addition and multiplication. The additive identity is 0, the additive inverse of x + iy is the complex number (−x) + i(−y), the multiplicative identity is 1 and the multiplicative inverse of the non–zero complex number x + iy is the complex number u + iv, where

$$u = \frac{x}{x^2 + y^2} \text{ and } v = \frac{-y}{x^2 + y^2}.$$

$$(\text{If } x + iy \neq 0, \text{ then } x \neq 0 \text{ or } y \neq 0, \text{ so } x^2 + y^2 \neq 0.)$$

we observe that addition and multiplication of complex numbers is performed just as for real numbers, replacing i^2 by −1, whenever it occurs.

A useful identity satisfied by complex numbers is

$$r^2 + s^2 = (r + is)(r - is).$$

This leads to a method of expressing the ratio of two complex numbers in the form x + iy, where x and y are real complex numbers.

$$\begin{aligned}\frac{x_1+iy_1}{x_2+iy_2} &= \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)}\\ &= \frac{(x_1x_2+y_1y_2)+i(-x_1y_2+y_1x_2)}{x_2^2+y_2^2}.\end{aligned}$$

The process is known as rationalization of the denominator.

3. Calculating With Complex Numbers

We can now do all the standard linear algebra calculations over the field of complex numbers – find the reduced row–echelon form of an matrix whose elements are complex numbers, solve systems of linear equations, find inverses and calculate determinants.

For example, solve the system

$$\begin{aligned}(1+i)z+(2-i)w &= 2+7i\\ 7z+(8-2i)w &= 4-9i.\end{aligned}$$

The coefficient determinant is

$$\begin{aligned}\begin{vmatrix}1+i & 2-i\\ 7 & 8-2i\end{vmatrix} &= (1+i)(8-2i)-7(2-i)\\ &= (8-2i)+i(8-2i)-14+7i\\ &= -4+13i\neq 0.\end{aligned}$$

Hence by Cramer's rule, there is a unique solution:

$$\begin{aligned}z &= \frac{\begin{vmatrix}2+7i & 2-i\\ 4-9i & 8-2i\end{vmatrix}}{-4+13i}\\ &= \frac{(2+7i)(8-2i)-(4-9i)(2-i)}{-4+13i}\\ &= \frac{2(8-2i)+(7i)(8-2i)-\{(4(2-i)-9i(2-i)\}}{-4+13i}\\ &= \frac{16-4i+56i-14i^2-\{8-4i-18i+9i^2\}}{-4+13i}\\ &= \frac{31+74i}{-4+13i}\\ &= \frac{(31+74i)(-4-13i)}{(-4)^2+13^2}\\ &= \frac{838-699i}{(-4)^2+13^2}\\ &= \frac{838}{185}-\frac{699}{185}i\end{aligned}$$

and similarly $w=\frac{-698}{185}+\frac{229}{185}i$.

An important property enjoyed by complex numbers is that every complex number has a square root:

THEOREM.1

If w is a non–zero complex number, then the equation $z^2 = w$ has a solution $z \in C$.

Proof. Let $w=a+ib$, $a, b\in\mathbb{R}$.

Case 1. Suppose $b=0$. Then if $a>0$, $z=\sqrt{a}$ is a solution, while if $a<0$, $i\sqrt{-a}$ is a solution.

Case 2. Suppose $b\neq 0$. Let $z=x+iy$, $x, y\in\mathbb{R}$. Then the equation $z^2=w$ becomes

$$(x+iy)^2=x^2-y^2+2xyi=a+ib,$$

so equating real and imaginary parts gives

$x^2 - y^2 = a$ and $2xy = b.$

Hence $x \neq 0$ and $y = b/(2x)$. Consequently

$$x^2 - \left(\frac{b}{2x}\right)^2 = a,$$

so $4x^4 - 4ax^2 - b^2 = 0$ and $4(x^2)^2 - 4a(x^2) - b^2 = 0$. Hence

$$x^2 = \frac{4a \pm \sqrt{16a^2 + 16b^2}}{8} = \frac{a \pm \sqrt{a^2 + b^2}}{2}.$$

However $x^2 > 0$, so we must take the + sign, as $a - \sqrt{a^2 + b^2} < 0$. Hence

$$x^2 = \frac{a + \sqrt{a^2 + b^2}}{2}, \quad x = \pm\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}}.$$

Then y is determined by y = b/(2x).

EXAMPLE 1 Solve the equation $z^2 = 1 + i$.

Solution. Put z = x + iy. Then the equation becomes

$(x + iy)^2 = x^2 - y^2 + 2xyi = 1 + i,$

so equating real and imaginary parts gives

$x^2 - y^2 = 1$ and $2xy = 1$.

Hence $x \neq 0$ and $y = 1/(2x)$. Consequently

$$x^2 - \left(\frac{1}{2x}\right)^2 = 1,$$

so $4x^4 - 4x^2 - 1 = 0$. Hence

$$x^2 = \frac{4 \pm \sqrt{16 + 16}}{8} = \frac{1 \pm \sqrt{2}}{2}.$$

Hence

$$x^2 = \frac{1 + \sqrt{2}}{2} \quad \text{and} \quad x = \pm\sqrt{\frac{1 + \sqrt{2}}{2}}.$$

Then

$$y = \frac{1}{2x} = \pm\frac{1}{\sqrt{2}\sqrt{1 + \sqrt{2}}}.$$

Hence the solutions are

$$z = \pm\left(\sqrt{\frac{1 + \sqrt{2}}{2}} + \frac{i}{\sqrt{2}\sqrt{1 + \sqrt{2}}}\right).$$

EXAMPLE.2 Solve the equation $z^2 + (\sqrt{3} + i)z + 1 = 0.$

Solution. Because every complex number has a square root, the familiar formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the solution of the general quadratic equation $az^2 + bz + c = 0$ can be used, where now $a(\neq 0), b, c \in \mathbb{C}.$ Hence

$$\begin{aligned} z &= \frac{-(\sqrt{3} + i) \pm \sqrt{(\sqrt{3} + i)^2 - 4}}{2} \\ &= \frac{-(\sqrt{3} + i) \pm \sqrt{(3 + 2\sqrt{3}i - 1) - 4}}{2} \\ &= \frac{-(\sqrt{3} + i) \pm \sqrt{-2 + 2\sqrt{3}i}}{2}. \end{aligned}$$

Now we have to solve $w^2 = -2 + 2\sqrt{3}i$. Put $w = x + iy$. Then $w^2 = x^2 - y^2 + 2xyi = -2 + 2\sqrt{3}i$ and equating real and imaginary parts gives $x^2 - y^2 = -2$ and $2xy = 2\sqrt{3}$. Hence $y = \sqrt{3}/x$ and so $x^2 - 3/x^2 = -2$. So $x^4 + 2x^2 - 3 = 0$ and $(x^2+3)(x^2-1) = 0$. Hence $x^2 - 1 = 0$ and $x = \pm 1$. Then $y = \pm\sqrt{3}$. Hence $(1+\sqrt{3}i)^2 = -2 + 2\sqrt{3}i$ and the formula for z now becomes

$$\begin{aligned} z &= \frac{-\sqrt{3} - i \pm (1+\sqrt{3}i)}{2} \\ &= \frac{1-\sqrt{3}+(1+\sqrt{3})i}{2} \quad \text{or} \quad \frac{-1-\sqrt{3}-(1+\sqrt{3})i}{2}. \end{aligned}$$

EXAMPLE.3 Find the cube roots of 1.

Solution. We have to solve the equation $z^3 = 1$, or $z^3 - 1 = 0$. Now $z^3 - 1 = (z - 1)(z^2 + z + 1)$. So $z^3 - 1 = 0 \Rightarrow z - 1 = 0$ or $z^2 + z + 1 = 0$. But

$$z^2 + z + 1 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1^2 - 4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

So there are 3 cube roots of 1, namely 1 and $(-1 \pm \sqrt{3}i)/2$.

We state the next theorem without proof. It states that every non– constant polynomial with complex number coefficients has a root in the field of complex numbers.

4. Proposal for better integration of complex numbers

In the introductory part of this paper, we addressed complex numbers, algebraic structures, and Euclid's plane geometry studied in many separate teaching disciplines at many of the faculties that prepare future teachers of mathematics. At that, as we can see from the aims that need to be achieved while studying the mentioned content we have minimal integration of content studied with complex numbers and Euclid's plane geometry. Further in this paper we will present program content that can significantly improve integration of teaching, and also improve the training of future mathematics teachers. Also, we can achieve the last with the introduction of a new teaching course Geometry of complex numbers, as well as with the study of the mentioned content within the existing courses in elementary mathematics found at the majority of faculties that prepare mathematics teachers.

Faculties that train future mathematics teachers, without any exception, have the subject geometry, most frequently, with the following syllabus:

- Introduction to geometry: basic elements and basic assertions, axioms of incidence, order, congruence, continuity and their consequences, and axiom of parallelism,
- Congruence: isometric transformations, congruence of figures, congruence of line segments, congruence of angles, congruence of triangles, angles of a transversal, (sum of angles in a triangle), inequality of triangles, rectangle, parallelogram, midsegments of a triangle, important points of a triangle,
- Application of congruence: application of circle congruence, central and inscribed angle, tangential rectangle, cyclic quadrilateral, normal lines and planes, dihedral, orthogonality of planes, angle between a line and a plane, angle between skew lines,
- Isometric plane transformations: direct and indirect transformations, reflection symmetry, central rotation, central symmetry, translation, slide symmetry, classification of plane isometries,
- Similarities: proportion of segments (Thales' theorem), homothety, similarity transformations (similar figures), similarity of triangles, Appolonius circle, degree of a point in relation to a circle, selected geometry theorems, and
- Inversion: definition and basic properties of inversion, Appolonius problems.

On other hand, within the subject of Complex analysis, the following content are frequently moved:

- Complex numbers (basic properties) algebraic form of complex numbers and complex conjugate, trigonometric form of a complex number, roots of complex numbers and exponential form of complex numbers, and
- Geometrical interpretation of a complex number, extended complex plane and Riemann's interpretation of complex number.

Later follows the study of the topological properties of the set of complex numbers, then moves on to differentiability and integrability, and then at a certain point there is the study of conformal translation. Clearly, this positioning of study of complex numbers does not allow integration of teaching with Euclid's geometry, which ca be significantly improved if the following contents are studied together with complex numbers:

- Equation of a line, self-conjugate equation of a line and distance from a point to a line,
- Equation of a circle, self-conjugate equation of a circle,
- Direct similarities, movements, homothety, indirect similarities and inversion,
- Möbius transformation, geometric properties of Möbius' transformation,

- Central and inscribed angle of a circle, radical axis and radical center,
- Important points of a triangle, right triangle, area of a triangle, circumcircle and incircle of a triangle,
- Euler line and Euler circle, theorems of: Manelaus, Desargues, Pascal, Ceva, Stewart and Ptolemy, Simson line, etc.

5. Conclusion

Integrating mathematics teaching with other teaching disciplines is of great importance and interest, however, it cannot be successfully achieved without firstly providing strong interdisciplinary integration of mathematics itself. Previously we discussed how we can achieve this in secondary education using complex numbers. However, this is not possible if we don not make the necessary changes in the education of future mathematics teachers. It is very important to guide the education of teachers towards their future profession, which means that the same should not be burdened with content that will not be used while they teach students, and at the same time be deficient in advanced knowledge that will become their daily routine (number theory, trigonometry, complex numbers and similar).

References

1. Malcheski, R. (2012). Didactic of mathematics teaching (basic part), FON University, Skopje
2. Malcheski, R., Anevska, K., Malcheski, S. (2014). Complex number – their application in geometry , ALFA '94,
3. Skopje Krtinić, Đ. (2012). Hight school Maтhematical competitions 2007-2011, DMS, Belgrade
4. Gogovska, V., Malcheski, R. (2012). Improvement intra-disciplinary integration of mathematics instruction, Elsevier, Procedia - Social and Behavioral Sciences, 46, 5420-5424
5. Penrose, Roger (2005), The Road to Reality: A Complete Guide to the Laws of the Universe, Alfred A. Knopf, ISBN 978-0-679-45443-4
6. Derbyshire, John (2006), Unknown Quantity: A Real and Imaginary History of Algebra, Joseph Henry Press, ISBN 978-0-309-09657-7
7. Needham, Tristan (1997), Visual Complex Analysis, Clarendon Press, ISBN 978-0-19-853447-1
8. Katz, Victor J. (2004), A History of Mathematics, Brief Version, Addison-Wesley, ISBN 978-0-321-16193-2
9. Nahin, Paul J. (1998), An Imaginary Tale: The Story of {\displaystyle \scriptstyle {\sqrt {-1}}}\scriptstyle {\sqrt {-1}}, Princeton University Press, ISBN 978-0-691-02795-1
10. Ebbinghaus, H. D.; Hermes, H.; Hirzebruch, F.; Koecher, M.; Mainzer, K.; Neukirch, J.; Prestel, A.; Remmert, R. (1991), Numbers (hardcover ed.), Springer, ISBN 978-0-387-97497-2