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# A Study of Graph Value Functions in Domination Theory

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**ABSTRACT**

The Y -domination number of a graph for a given number set Y was introduced by D.W. Bange, A. E. Barkauskas, L. H. Host and P. J. Slater as a generalization of the domination number of a graph. It is defined using the concept of a Y -dominating function. The total dominating graph Dt(G) of G is the graph with the vertex set V∪S in which two vertices u and v are adjacent if u ∈ V and v is a minimal total dominating set of G containing u. In this paper, some properties of this new graph are obtained. Also characterizations are given for graphs.

## 1. Introduction

All graphs considered here are finite, undirected without loops and multiple edges. Any undefined term in this paper may be found in [1, 2]. Let G = (V, E) be a graph. A set D of vertices in a graph G is called a dominating set of G if every vertex in V – D is adjacent to some vertex in D. The domination number γ(G) of G is the minimum cardinality of a dominating set in G. Recently several domination parameters are given in the books by Kulli in [2, 3, 4]. A set D of vertices in G is a total dominating set of G if every vertex of G is adjacent to some vertex in D. The total domination number γt(G) of G is the minimum cardinality of a total dominating set of G. A total dominating set D of G is minimal if for any vertex v ∈ D, D –{v} is not a total dominating set of G. We note that any graph G without isolated vertices has a total dominating set. Thus we consider only graphs without isolated vertices. The total minimal dominating graph Mt(G) of a graph G is the intersection graph defined on the family of all minimal total dominating sets of vertices of G. This concept was introduced by Kulli and Iyer in [5]. The common minimal total dominating graph CDt(G) of a graph G is the graph with same vertex set as G with two vertices in CDt(G) adjacent if there exists a minimal total dominating set in G containing them. This concept was introduced by Kulli in [6]. The dominating graph D(G) of a graph G is the graph with the vertex set V ∪ S where S is the set of all minimal dominating sets of G in which two vertices u and v are adjacent if u ∈ V and v is a minimal dominating set in G containing u. This concept wasintroduced by Kulli et al in [7]. Many other graph valued functions in domination theory were studied,

## 2. Dominating functions of graphs

A subset D of the vertex set V (G) of a graph G is called dominating in G, if for each vertex x ∈ V (G) − D there exists a vertex y ∈ D adjacent to x. The minimum number of vertices of a dominating set in G is called the domination number of G and denoted by γ(G). This well-known concept can be defined in another way, using domination functions. We will speak about functions f which map V (G) into some set of numbers. If S ⊆ V (G), then we denote f(S) = x∈S f(x). If x ∈ V (G), then by N[x] we denote the closed neighbourhood of x in G, i.e. the set consisting of x and of all vertices which are adjacent to x in G. Besides, we will also consider the open neighbourhood N(x) = N[x] − {x}. Now we can formulate the alternative definition of the domination number.

A function f : V (G) → {0, 1} is called a dominating function of G, if f(N[x]) 1 for each x ∈ V (G). The minimum sum f(V (G)) = x∈V (G) f(x) taken over all dominating functions f of G is called the domination number of G and denoted by γ(G).It is evident that these two definitions are equivalent. Namely, if D is a dominating set in G, then the function f defined so that f(x) = 1 for x ∈ D and f(x) = 0 for x ∈ V (G) − D is a dominating function of G. Conversely, if f is a dominating function of G, then the set D = {x ∈ V (G); f(x)=1} is a dominating set in D.

The concept of a dominating function and obviously also the related concept of the domination number were generalized by some authors in such a way that the set of values {0, 1} was replaced by another number set. In [1] the signed dominating function and the signed domination number were defined by replacing the set {0, 1} by {−1, 1} and in [2] the minus dominating function and the minus domination number were defined by using the set {−1, 0, 1}. The fractional dominating function and the fractional domination number were introduced in [3] by using the set of real numbers. The most general case is the Y -dominating function and the Y -domination number, where a quite arbitrary set Y of values of f is used [4]. Therefore, following [4], a function f : V (G) → Y , where Y is a given set of numbers, is called a Y -dominating function of G, if f(N[x]) 1 for each x ∈ V (G). The minimum of f(V (G)) taken over all Y -dominating functions f of G is called the Y -dominating number of G and is denoted by γY (G). We will not treat the domination is such a general way. We restrict our considerations to natural generalizations of the set {0, 1}, namely to two-element number sets {0, t}, where t is a positive real number.

**Proposition 1.**

Let Y = {0, t}, where t is a positive real number. Let G be a graph. The Y -domination number γY (G) of G is defined and at least one Y - dominating function of G exists if and only if δ(G) 1/t − 1, where δ(G) denotes the minimum degree of a vertex of G.

 

Let f be a function which maps V (G) into the set of real numbers and let x ∈ V (G). The vertex set x will be called a zero vertex of f, if f(x) = 0. The following theorem enables us to restrict our consideration to numbers t which are inverses of positive integers.

**Theorem 1.** Let t be a positive real number, let G be a graph with δ(G) 1/t−1. Let k =

1/tand Y1 = {0, t}, Y2 = {0, 1/k}. Then γY1 (G) = ktγY2 (G) and there exists a one-to-one correspondence between Y1-dominating functions of G and Y2- dominating functions of G such that the corresponding functions have the same set of zero vertices

**Proof:** Let *f* : V (G) → Y1, g : V (G) → Y2 and suppose that f,g have the same set of zero vertices. Then $f(x) = ktg(x)$ and also $f(N[x]) = ktg(N[x])$ f

each $x \in V(G)$. Suppose that $g$ is a $Y_2$-dominating function of $G$: then $g(N[x]) \geqslant 1$ for each $x \in V(G)$. Evidently $kt \geqslant 1$ and thus $f(N[x]) \geqslant g(N[x]) \geqslant 1$ for each $x \in V(G)$ and $f$ is a $Y_1$-dominating function of $G$. Now suppose that $g$ is not a $Y_2$-dominating function of $G$. There exists $x \in V(G)$ such that $g(N[x]) < 1$. If $k = 1$, then $g(N[x])$ must be a non-negative integer and therefore $g(N[x]) = 0$. This is possible only if $g(y) = 0$ for each $y \in N[x]$. But then also $f(y) = 0$ for each $y \in N[x]$ and $f(N[x]) = 0$; the function $f$ is not a $Y_1$-dominating function of $G$. If $k \geqslant 2$, then the number of vertices of $N[x]$ which are not zero vertices of $g$ is at most $k-1$. But these vertices are exactly those vertices which are not zero vertices of $f$. We have $f(N[x]) \leqslant (k-1)t$. Evidently $1/t > k-1$ and thus $f(N[x]) \leqslant (k-1)t < 1$; the function $f$ is not a $Y_1$-dominating function of $G$. If $g_0$ is a minimal (i.e. with the minimum sum on $V(G)$) $Y_2$-dominating function, then the corresponding function $f_0$ is a minimal $Y_1$-dominating function. We have $\gamma_{Y_1}(G) = \sum_{x \in V(G)} f_0(x) = \sum_{x \in V(G)} ktg_0(x) = kt \sum_{x \in V(G)} g_0(x) = kt\gamma_{Y_2}(G)$. □

For each positive integer $k$ we denote $Y(k) = \{0, 1/k\}$ and $\gamma(k, G) = \gamma_{Y(k)}G$. From Proposition 1 we have the following corollary.

**Total dominating graphs**

The definition of dominating graph of a graph inspired us to define the following graph valued function in domination theory.

**Definition 1.** Let G = (V, E) be a graph. Let S be the set of all minimal total dominating sets of G. The total dominating graph Dt(G) of G is the graph with the vertex set V ∪ S in which two vertices u and v are adjacent if u ∈ V and v is a minimal total dominating set of G containing u.

**Example 2.** In Figure 1, a graph G and its total dominating graph Dt(G) are shown.

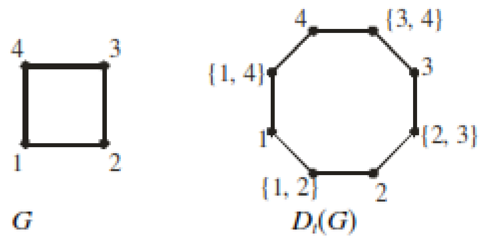


**Figure 1**

**Proposition 3.** If G has a vertex which does not lie in any minimal total dominating set, then Dt(G) is disconnected.

**Proof:** Let u be a vertex of a graph G. If u does not lie in any minimal total dominating set, then u is an isolated vertex in Dt(G). Hence Dt(G) is disconnected.

**Theorem 4.** If G is a graph without isolated vertices, then Dt(G) is bipartite.

**Proof:** By definition, no two vertices corresponding to vertices of G in Dt(G) are adjacent and also no two vertices corresponding to minimal total dominating sets of G in Dt(G) are adjacent. Thus Dt(G) has no odd cycles. By Theorem A, Dt(G) is bipartite.

**Theorem 5.** The total dominating graph Dt(G) of G is complete bipartite if and only if G = mK2, m ≥ 1.

**Proof:** Suppose Dt(G) is complete bipartite. Clearly V(Dt(G)) = V1 ∪ V2, where V1 is the set of all vertices of G and V2 is the set of all minimal total dominating sets of G. We now prove that G = mK2, m ≥ 1. On the contrary, assume G ≠ mK2. Then there exists a component G1 in G which is not K2. Let v be a vertex of G1. Then v ∈ G. We consider the following two cases:

**Case 1.** Suppose v ∉ D, where D is any minimal total dominating set in G. Then the corresponding vertex of v is an isolated vertex in Dt(G). It implies that the corresponding vertices of D and v are not adjacent in Dt(G). Thus Dt(G) is not complete bipartite, which is a contradiction.

**Case 2.** Suppose there exist two minimal total dominating sets D1 and D2 such that v ∈ D1 and v ∉ D2. Thus the corresponding vertices of v and D2 are not adjacent in Dt(G). Hence Dt(G) is not complete bipartite, which is a contradiction.

From Case 1 and Case 2, we conclude that every component of G is K2. Thus G = mK2, m ≥ 1.

Conversely, suppose G = mK2, m ≥ 1. Then there exists exactly one minimal total dominating set containing all vertices of G. Then |V(Dt(G) | = 2m +1. Thus by definition Dt(G) = K1, 2m and hence Dt(G) is complete bipartite.

**Theorem 5.** *Let $k, m, n$ be positive integers, $k-1 \leqslant m \leqslant n$. If $k < m$, then $\gamma(k, K_{m,n}) = 2$. If $m = k-1$, then $\gamma(k, K_{m,n}) = (m+n)/k = (k+n-1)/k$. If $m = k$, then $\gamma(k, K_{m,n}) = 2 - 1/k$.*

Proof. Let $k < m$. Let $A, B$ be the bipartition classes of $K$, $|A| = m$, $|B| = n$. For each vertex $x \in A$, its open neighbourhood satisfies $N(x) \subseteq B$. As $N[x] = \{x\} \cup N(x)$ and $f(N[x]) \geqslant 1$ for a $Y(k)$-dominating function $f$, there are at least $k-1$ vertices $y \in N(x) \subseteq A$ which are in $V^+$. If moreover $f(x) = 0$, then there are at least $k$ such vertices. Therefore either $f(x) = 1/k$ for all $x \in A$ and $f(y) = 1/k$ for at least $k-1$ vertices of $B$, or $f(y) = 1/k$ for at least $k$ vertices of $B$. In the former case $f(V(K_{m,n})) \geqslant (m+k-1)/k \geqslant 2$. In the latter case analogously either $f(x) = 1/k$ for all $x \in B$ and $f(y) = 1/k$ for at least $k-1$ vertices of $A$, or $f(y) = 1/k$ for at least $k$ vertices of $A$. In both these cases again $f(V(K_{m,n})) \geqslant 2$. A function $f$ which assigns $1/k$ to exactly $k$ vertices of $A$ and to exactly $k$ vertices of $B$ has $f(V(K_{m,n})) = 2$.

Now suppose $m = k-1$. Then $|A| = k-1$. Let $x \in B$ and again let $f$ be a $Y(k)$-dominating function of $K_{m,n}$. The set $N[x]$ has exactly $k$ vertices and thus $f(x) = 1/k$ for each $y \in N[x]$. This means that $f(y) = 1/k$ for each $y \in A$ and also $f(x) = 1/k$. As $x$ is an arbitrary vertex of $B$, we have $f(x) = 1/k$ for all $x \in V(K_{m,n})$ and $f(V(K_{m,n})) = (k-1+n)/k$. Another $Y(k)$-dominating function does not exist and thus $\gamma(k, K_{m,n}) = (k-1+n)/k$.

Finally, let $k = m$. If $f$ is a $Y(k)$-dominating function, then either $f(x) = 1/k$ for each $x \in A$ and for at least $k-1$ vertices $x$ of $B$, or $f(x) = 1/k$ for exactly $k-1$ vertices of $A$ and all vertices $x \in B$. In the former case $f(V(K_{m,n})) \geqslant (2k-1)/k = 2-1/k$, in the latter case $f(V(K_{m,n})) \geqslant (k-1+n)/k \geqslant (2k-1)/k = 2-1/k$. If $f$ assigns the value $1/k$ to all vertices of $A$ and to exactly $k-1$ vertices of $B$, then $f(V(K_{m,n})) = 2-1/k$, therefore $\gamma(k, K_{m,n}) = 2-1/k$. □

By the symbol $G \oplus H$ we denote the Zykov sum of graphs $G$ and $H$, i.e. the graph obtained from vertex-disjoint graphs $G$ and $H$ by joining all vertices of $G$ with all vertices of $H$ by edges.

**Theorem 6.** *Let $k, q$ be positive integers, let $G, H$ be two graphs such that $\gamma(k, G), \gamma(k, H)$ are defined and $q \leqslant 1 + \min(\gamma(k, G), \gamma(k, H))$. Then $\gamma(kq, G \oplus H) \leqslant (\gamma(k, G) + \gamma(k, H))/q$.*

Proof. Let $g$ and $h$ be minimal $Y(k)$-dominating functions of $G$ and $H$, respectively. Let $f \colon V(G) \cup V(H) \to Y(kq)$ be defined so that $f(x) = g(x)/q$ for $x \in V(G)$ and $f(x) = h(x)/q$ for $x \in V(H)$. Consider $x \in V(G)$. The closed neighbourhood of $x$ in $G \oplus H$ is the disjoint union of the closed neighbourhood of $x$ in $G$ and of $V(H)$. The sum of values of $f$ over the closed neighbourhood of $x$ in $G$ is at least $1/q$, its sum over $V(H)$ is at least $\gamma(k, H)/q$. It follows from the assumption that $1/q + \gamma(k, H)/q \geqslant 1$. For $x \in V(H)$ this may be proved quite analogously. Therefore $f$ is a $Y(kq)$-dominating function of $G \oplus H$. This implies the assertions. □

Theorem 7. If $S_{m,n}$, $1 \le m \le n$, is a double star, then

$$D_t(S_{m,n}) = (m+n)\, K_1 \cup K_{1,2}.$$

**Proof:** Let Sm, n be a double star, 1≤m≤n and u and v be central vertices of Sm, n. Then Sm, n has exactly one minimal total dominating set D containing the central vertices u and v of Sm, n. Then D = {u, v}. Thus the vertex set of Dt(Sm, n) in V∪D, where V is a vertex set of Sm, n, and hence Dt(Sm,n) has m+n+2+1 vertices. The corresponding vertices of D and u are adjacent and also the corresponding vertices of D and v are adjacent in Dt(Sm, n) and all other vertices of Dt(Sm, n) are isolated vertices. Thus Dt(Sm, n) is disconnected and

$$D_t(S_{m,n}) = (m+n)K_1 \cup K_2.$$

**Theorem 8.** Let G be a nontrivial connected graph. Let S(G) be the subdivision graph of G. The graphs Dt(G) and S(G) are isomorphic if and only if every pair of vertices forms a minimal total dominating set of G.

**Proof:** Let G be a nontrivial connected graph. Suppose Dt(G) = S(G). Since G is connected, S(G) is connected. For each edge ei=uivi of G, wi is a new vertex such that uiwi and wivi are edges of S(G). Since Dt(G)= S(G), it implies that every pair of vertices ui ,vi forms a minimal total dominating set of G.

Conversely, suppose every pair of vertices of G forms a minimal total dominating set of G. Then they are adjacent in G. Clearly for each minimal total dominating set D of G, the corresponding vertex of D in Dt(G) is adjacent with exactly two vertices and hence we see that Dt(G) = S(G).

## 3. Conclusion

A majority dominating function on the vertex set of a graph G = (V; E) is a function g : V → {1; −1} such that g(N[v]) ¿ 1 for at least half of the vertices v in V. The weight of a majority dominating function is denoted as g(V) and is  g(v) over all v in V. The majority domination number of a graph is the minimum possible weight of a majority dominating function, and is denoted. We determine the majority domination numbers of certain families of graphs. Moreover, we show that the decision problem corresponding to computing the majority domination number of an arbitrary disjoint union of complete graphs is NP-complete.

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