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Study on Different Orders of Bessel functions and Laplace Transform Solution of Zero-Order Bessel Equation

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ABSTRACT

To locate the zero request Bessel functions we apply, the Laplace change to the zero request Bessel equation and express the arrangement of the changed equation in infinite series structure in the change variable utilizing the binomial hypothesis; on reversing the infinite series in the change variable term-by-term we acquire the infinite series type of the answer for the zero-request Bessel equation, that is, the zero-request Bessel work.

1. Introduction

While uncommon sorts of what might later be known as Bessel functions were contemplated by Euler, Lagrange, and the Bernoullis, the Bessel functions were first utilized by F. W. Bessel to depict three body movement, with the Bessel functions showing up in the series extension on planetary bother. This paper displays the Bessel functions as emerging from the arrangement of a differential equation; an equation which shows up much of the time in applications and answers for physical circumstances. As often as possible, the way to taking care of such issues is to perceive the type of this equation, consequently permitting work of the Bessel functions as arrangements. The subject of Bessel Functions and applications is a rich subject; by the by, because of reality limitations.

The significance of the Bessel equation (in its different structures) and the subsequent Bessel functions in pragmatic applications of mathematical material science can scarcely be overstated and it follows, normally, that an introduction to Bessel functions must have a significant impact of the mathematical training any researcher or engineer. In this paper we propose an

introduction to Bessel functions j (x) n through a hybrid approach to the solution of the Bessel equation of integral-order, that is

$$x^2\frac{d^2 j_n(x)}{dx^2}+x\frac{dj_n(x)}{dx}+(x^2-n^2)j_n(x)=0 \qquad (1)$$

With n (at first) a non-negative integer

This hybrid methodology consolidates the assurance of 'raising' and 'bringing down' differential operators (from the factorization of Bessel's equation, conspicuous as standard repeat relations for Bessel functions) with the arrangement of the zero-order Bessel equation (0=n) through the utilization of the Laplace change. The arrangement of the zero order Bessel equation by means of the Laplace change can be communicated in infinite series structure (through a legal utilization of the Binomial theorem; see underneath) and afterward, utilizing, recursively, the 'raising' ladder-administrator (and standard outcomes on the manipulation of infinite series) the Bessel work answer for the indispensable order Bessel equation (1) can be resolved (through mathematical induction, basically). En route, fundamental outcomes in the manipulation of infinite series are cited (sourced) as required.

2. The Bessel Equation

Bessel's equation is a second order differential equation of the form

$$x^2y''+xy'+(x^2-\nu^2)y=0 \qquad (2)$$

By re-writing this equation as:

$$x(xy')'+(x^2-\nu^2)y=0 \qquad (3)$$

and employing the use of a generalized power series, we re-write the terms of (3) in terms of the series:

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} a_n x^{n+s}\\
y' &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s-1}\\
xy' &= \sum_{n=0}^{\infty} a_n (n+s) x^{n+s}\\
(xy')' &= \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s-1}\\
(xy')' &= \sum_{n=0}^{\infty} a_n (n+s)^2 x^{n+s}
\end{aligned}$$

When the coefficients of the powers of x are organized, we find that the coefficient on x^s gives the indicial equation $s^2 - v^2 = 0$, =⇒ s = ±v, and we develop the general formula for the coefficient on the x^{s+n} term:

$$a_n=-\frac{a_{n-2}}{(n+s)^2-\nu^2} \qquad (4)$$

In the case s = v:

$$a_n=-\frac{a_{n-2}}{n(n+2\nu)} \qquad (5)$$

and since $a_1 = 0$, $a_n = 0$ for all n = odd integers. Coefficients for even powers of n are found:

$$a_{2n} = -\frac{a_{2n-2}}{2^2 n(n+\nu)} \quad (6)$$

Recalling that for the gamma function:

$$\Gamma(\nu+2) = (\nu+1)\Gamma(\nu+1),$$

$$\Gamma(\nu+3) = (\nu+2)\Gamma(\nu+2) = (\nu+2)(\nu+1)\Gamma(\nu+1),$$

we can write the coefficients:

$$a_2 = -\frac{a_0}{2^2(1+\nu)} = -\frac{\Gamma(1+\nu)}{2^2\Gamma(2+\nu)}$$

$$a_{2n} = -\frac{a_0\Gamma(1+\nu)}{n!2^{2n}\Gamma(n+1+\nu)}$$

Which allows us to write the terms of the series

$$y = J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n+1)\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n+\nu} \quad (7)$$

Where $J_v(x)$ is the Bessel function of the first kind, order v. The first five Bessel functions of this kind are shown in figure 1

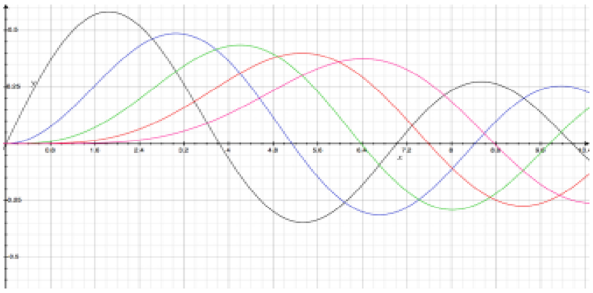

FIG. 1: The Bessel Functions of orders v = 0 to v = 5

3. Different Orders of Bessel functions

In the previous area, the type of Bessel functions were acquired are known as "Bessel functions of the first kind". Various types of Bessel functions are gotten with negative estimations of v, or with complex contentions. This segment quickly investigates these various types of functions

Neumann Functions

Bessel functions of the second kind are known as Neumann functions, and are developed as a linear combination of Bessel functions of the first order described:

$$N_\nu(x) = \frac{cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{sin\nu\pi} \quad (8)$$

For integral values of v, the expression of $N_v(x)$ has an indeterminate form, and $N_\nu(x)|_{x=0} = \pm\infty$ Nevertheless the limit of this function for the expression for N_v is valid for any value of v, allowing the general solution to Bessel's equation to be written:

$$y = AJ_\nu(x) + BN_\nu(x) \quad (9)$$

With A and B as subjective constants decided from limit conditions. Bessel functions of the first and second kind are the most ordinarily discovered types of the Bessel work in applications. Numerous applications in hydrodynamics, versatility, and oscillatory frameworks have arrangements that depend on the Bessel functions. One such model is that of a uniform thickness chain fixed toward one side experiencing little motions. The differential equation of this circumstance is:

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \frac{k^2u}{z} = 0 \quad (10)$$

Where z references a point on the chain $k^2 = \frac{p^2}{g}$, with p as the frequency of small oscillations at that point, and g the gravitational constant of acceleration Eq. (10) is a form of eq. (2), and solution is:

$$u = AJ_0(2kz^{\frac{1}{2}}) + BY_0(2kz^{\frac{1}{2}}), \quad (11)$$

Where the A and B are determined by the boundary conditions

Modified Bessel Functions

Modified Bessel functions are found as solutions to the modified Bessel equation

$$x^2y'' + xy' - (x^2 - \nu^2)y = 0 \quad (12)$$

which transforms into eq. (2) when x is replaced with ix. However, this leaves the general solution of eq. (2) a complex function of x. To avoid dealing with complex solutions in practical applications , the solutions to (12) are expressed in the form:

$$I_\nu(x) = e^{\frac{\nu\pi i}{2}} J_\nu(xe^{\frac{i\pi}{2}}) \quad (13)$$

The $I_v(x)$ are a set of functions known as the modified Bessel functions of the first kind. The general solution of the modified Bessel function is expressed as a combination of $I_v(x)$ and a function $I_{-v}(x)$:

$$y = AI_{-\nu}(x) - BI_\nu(x) \quad (14)$$

where again A and B are determined from the boundary conditions.

A solution for non-integer orders of v is found:

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{sin\nu\pi} \quad (15)$$

The functions $K_v(x)$ are known as modified Bessel functions of the second kind. A plot of the Neumann Functions ($N_v(x)$) and Modified Bessel functions ($I_v(x)$)is shown in figure (2).

A plot of the Modified Second Kind functions ($K_n(x)$) is shown in fig. (3). Modified Bessel functions appear less frequently in applications, but can be found in transmission line studies, non-uniform beams, and the statistical treatment of a relativistic gas in statistical mechanics.

Zeroes of Bessel Functions

The zeroes of Bessel functions are of great importance in applications. The zeroes, or roots, of the Bessel functions are the values of x where value of the Bessel function goes to zero ($J_v(x) = 0$). Frequently, the zeroes are found in tabulated formats, as they must the be numerically evaluated. Bessel function's of the first

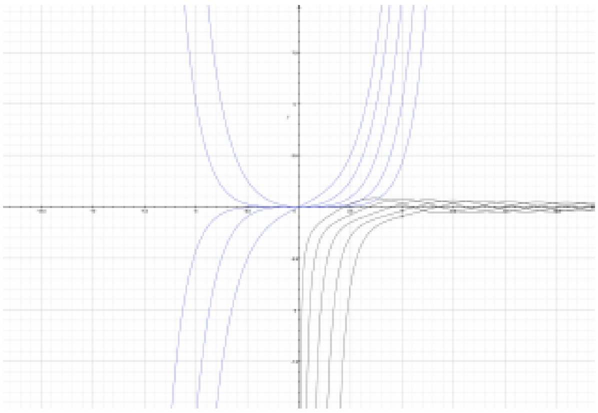

FIG. 2: The Neumann Functions (black) and the Modified Bessel Functions (blue) for integer orders v = 0 to v = 5

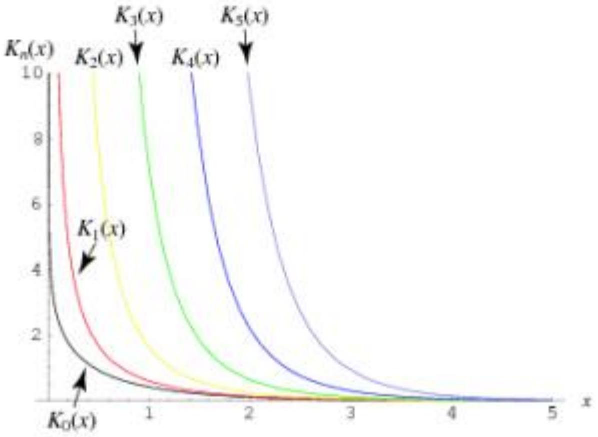

FIG. 3: The Modified Bessel Functions of the second kind for orders v = 0 to v = 5

and second kind have an infinite number of zeros as the value of x goes to ∞. The zeroes of the functions can be seen in the crossing points of the graphs in figure (1), and figure (2). The modified Bessel functions of the first kind (Iv(x)) have only one zero at the point x = 0, and the modified Bessel equations of the second kind (Kv(x)) functions do not have zeroes.

Bessel function zeros are exploited in frequency modulated (FM) radio transmission. FM transmission is mathematically represented by a harmonic distribution of a sine wave carrier modulated by a sine wave signal which can be represented with Bessel Functions. The carrier or sideband frequencies disappear when the modulation index (the peak frequency deviation divided by the modulation frequency) is equal to the zero crossing of the function for the n th sideband.

4. Laplace Transform Solution of Zero-Order Bessel Equation

Consider, now, the Bessel equation of zero-order, that is

$$x\frac{d^2 j_n(x)}{dx^2}+\frac{dj_n(x)}{dx}+xj_n(x)=0 \quad \textbf{(16)}$$

We will understand (16) utilizing the Laplace change (the strategy is outstanding and surely knew and revamp the arrangement, utilizing the Binomial theorem, as an infinite series in the changed variable; at that point, we will reverse the infinite series in the changed variable (term-by-term) to acquire the typical infinite series answer for as far as the first variable x.

First, we introduce the Laplace transform $L[f(x)]\equiv F(s)$ of the function f (x) as the usual improper integral

$$L[f(x)]\equiv F(s)=\int_0^\infty e^{-sx}f(x)dx \quad \textbf{(17)}$$

and where we will make use of the following two well-known properties of the Laplace transform (with zero initial conditions, as we seek particular solutions)

$$L[f^{(k)}(x)]\equiv s^k F(s) \quad \textbf{(18)}$$

$$L[x^k f(x)]\equiv(-1)^k F^{(k)}(s)=(-1)^k(L[f(x)])^{(k)} \quad \textbf{(19)}$$

Where the superscripts imply differentiation with respect to either x or s From equations (18), we have the following result helping us to transform (16):

$$L[xf^{(2)}(x)]\equiv(-1)^1(L[f^{(2)}(x)])^{(1)}=(s^2F(s))^{(1)}$$

$$=-s^2F'(s)-2sF(s) \quad \textbf{(20)}$$

we Laplace transform (16), term-by term using (18) and (20), to get a differential equation for the transform $J_0(s)$ of $j_0(x)$:

$$(s^2+1)J_0'(s)+sJ_0(s)=0 \quad \textbf{(21)}$$

(returning to the 'dash' notation for differentiation) with the variable separable equation (21) having the particular solution

$$J_0(s)=(s^2+1)^{-\frac{1}{2}} \quad \textbf{(22)}$$

Now, to get a series solution to our transformed problem, we expand (22) via the Binomial theorem to get

$$J_0(s)=s^{-1}(1+s^{-2})^{-\frac{1}{2}}=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}s^{-(2k+1)} \quad \textbf{(23)}$$

which we may invert term-by-term using the result

$$L[x^k]=\int_0^\infty e^{-sx}x^k dx=\frac{k!}{s^{k+1}} \quad \textbf{(24)}$$

to get the infinite series expansion for J₀ (s) in the form

$$j_0(x)=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{x^{2k}}{(2k)!} \quad \textbf{(25)}$$

Finally, to get the infinite series expansion for J₀ (s) in the standard form, we evaluate the generalized Binomial coefficient to find that

$$\binom{-\frac{1}{2}}{k}\equiv\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\cdots\left(-\frac{1}{2}-(k-1)\right)}{k!}=\frac{(-1)^k}{2^k k!}(2k-1)!! \quad \textbf{(26)}$$

With

$$(2k-1)!!=(2k-1)(2k-3)(2k-5)\cdots5.3.1=\frac{(2k)!}{2^k k!} \quad \textbf{(27)}$$

where we have introduced in (27) the double factorial function 1)!!–(2k along with its main identity. So, substituting (27) into (26) and then (26) into (25) and re-arranging, we get the standard form of the infinite series expansion for J₀ (s)

$$j_0(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2} \right)^{2k} \qquad (28)$$

5. Conclusion

The assurance of the general-order Bessel work – at that point the Laplace change arrangement of a second-order linear conventional differential equation with variable coefficients and afterward, to crown this off in a manner of speaking, the introduction and manipulation of the infinite series portrayal of the zero-order and afterward the general-order Bessel work through the manipulation of the Binomial theorem. Laplace change arrangement of (16), first we experience a standard variable detachable differential equation in a novel circumstance and afterward we understand that specific conditions must be fulfilled and, in a study hall, talked about, for the term-by-term manipulation of the series (22) and (28) to be valid.

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