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# On clean rings and related generalizations

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**ABSTRACT**

A ring $R$ is called clean if every element of $R$ can be represented as a sum of an idempotent and a unit in $R$. These rings have been of immense interest in recent years. In this article, we give a brief survey of clean rings and its various generalizations.

## 1. Exchange rings and clean rings

A module $M_R$ is said to have an *exchange property* (see Crawley–Johnson [5]) if for every module $K_R$ and any two decompositions

$$K = M' \oplus P = \oplus_{i\in I}\ K_i$$

with $M' \cong M$, there exists submodules $K_i' \subset K_i$ such that

$$K = M' \oplus (\oplus_{i\in I}\ K_i').$$

And a module $M_R$ has the *finite exchange property* if the above condition is met whenever the indexing set $I$ is finite. Warfield in [29] called a ring $R$ an *exchange ring* if $R_R$ has the finite exchange property, equivalently, if it has the full exchange property. He proved that that the definition of an exchange ring is left-right symmetric.

The class of exchange rings includes von Neumann regular rings (the rings in which every principal right ideal is a direct summand of $R$ as a right $R$ module); the local rings (the rings which have a unique maximal right ideal) and also the semiperfect rings (the rings for which $R/J(R)$ is semisimple and idempotents of $R/J(R)$ can be lifted to $R$, where $J(R)$ denotes the Jacobson radical of the ring $R$).

Nicholson in his important paper [21] gave various characterizations of exchange rings. One of the important characterizations of exchange rings that he obtained was that a ring $R$ is an exchange ring if and only if for each $a \in R$, there exists an idempotent $e$ in $R$ such that $e \in Ra$ and $1-e \in R(1-a)$. He also provided a new subclass of exchange rings by observing that if a ring $R$ has the property that every element of $R$ is a sum of an idempotent and a unit in $R$, then $R$ is an exchange ring. He called a ring having this property a *clean ring.* Nicholson proved that if all idempotents of a ring $R$ are central, then $R$ is a clean ring if and only if it is an exchange ring. In particular, a commutative ring is an exchange ring if and only if it is a clean ring.

Since the introductory paper by Nicholson, clean rings have been an area of interest on their own for many researchers. Besides studying the properties of clean rings, researchers have subsequently been trying to find classes of rings that are clean and have also been trying to link various known classes of rings to clean rings. More recently, people have also been interested in studying various generalizations of clean rings.

It is easy to see that homomorphic image of a clean ring is clean and direct product of rings is clean if and only if each factor is clean. Also, it was observed by Nicholson [21] that a ring $R$ is clean if and only if $R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$. Han and Nicholson [16] observed that if $e$ is an idempotent in a ring $R$ such that both $eRe$ and $(1-e)R(1-e)$ are clean rings, then $R$ is also a clean ring. Using this, it was proved that any finite matrix ring $M_n(R)$ over a clean ring $R$ is also clean. But the question whether the corner ring of a clean ring is also clean, was left open for many years, which was finally settled in negative by Šter in [27].

Nicholson [21] proved that a commutative ring $R$ is a clean ring if and only if it is an exchange ring. Since their introduction, many authors have studied clean rings in various different contexts and as a result many interesting characterizations of clean rings have appeared over the years. McGovern in [19] collected a few characterizations in a single result, before stating that we recall a few definitions.

A commutative ring $R$ is called a *pm-ring* if every prime ideal of $R$ is contained in a unique maximal ideal of $R$. A ring $R$ is called a *Gelfand ring* if whenever $x+y=1$ in $R$, there exists $r,s \in R$ such that $(1+xr)(1+ys)=0$. Contessa [13] defined a ring $R$ to be *topologically-boolean* or *tb-ring* if for every pair of distinct maximal ideals of $R$, there is an idempotent belonging to exactly one of them.

An ideal $I$ of a ring $R$ is called *pure* if for any $r \in I$, there exists $s \in I$ such that $rs = r$. Vasconcelos [28] called a ring $\mathfrak{f}$-ring if every pure ideal of $R$ is generated by idempotents.

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For a commutative ring $R$, $Spec\ R$ denotes the collection of prime ideals of $R$ and $Max\ R$ denotes the collection of maximal ideals of $R$. For any $r \in R$, let $\mathcal{U}(r) = \{P \in Spec\ R : r \notin P\}$ and let $\mathcal{V}(r)$ denote the complement of $\mathcal{U}(r)$. Then the topology obtained by taking the set $\{\mathcal{U}(r)$: r$\in R\}$ as a base for the open sets is called *Zariski –toplogy* on $R$ and the topology on $Max\ R$ is simply the subspace topology induced from the zariski toplogy on $Spec\ R$. In particular, let, $U(r) = Max\ R \cap \mathcal{U}(r)$ and $V(r) = Max\ R \cap \mathcal{V}(r)$.We now give the collection of all characterizations of commutative clean rings. For topological terms used in the result, we refer to [15].

**Theorem 1.1** (McGovern [19]) *For a commutative ring $R$, the following are equivalent:*

1. *$R$ is an exchange ring.*
2. *$End(R)$ is an exchange ring.*
3. *Idempotents can be lifted modulo every ideal of $R$.*
4. *$R$ is a Gelfand ring and $Max\ R$ is zero-dimensional.*
5. *$R$ is a pm-ring and $Max\ R$ is zero-dimensional.*
6. *$Max\ R$ is a retract of $Spec\ R$ and $Max\ R$ is zero-dimensional.*
7. *$J(R)$ is weakly zero-dimensional.*
8. *$R$ is a clean ring.*
9. *$R/J(R)$ is clean and idempotents can be lifted modulo $J(R)$.*
10. *$R/Nil\ (R)$ is clean.*
11. *$R$ is a tb-ring, that is, for any pair of distinct maximal ideals there is an idempotent in exactly one of them.*
12. *For every $m, m' \in R$ with $1 = m + m'$, there is an idempotent $e$ such that $e \in Rm$ and $1 - e \in Rm'$.*
13. *The collection □ $= \{U(e): e\ an\ idempotent\}$ forms a base for the Zariski topology on $Max\ R$.*
14. *For each $r \in R$, there exists an idempotent $e$ such that $V(r) \subseteq U(e)$ and $V(a-1) \subseteq V(e)$.*
15. *$R$ is a pm-ring and an* ƒ- ring.

As mentioned earlier, every clean ring is an exchange ring and abelian (a ring all whose idempotents are central) exchange rings are clean also. But the question whether in general, the clean rings form a proper subclass of exchange rings remained open for quite long. Camillo and Yu in [8] proved that if $R$ is a clean ring in which $2$ is invertible, then every element of $R$ is a sum of a unit and a square root of $1$, in particular, every element of $R$ can be written as sum of two units. Also, there already existed an example due to Bergman of a von Neumann regular ring with 2 invertible in which not every element was a sum of finitely many units. As every von Neumann regular ring is an exchange ring (see Warfield [29]), this example of Bergman served as a tool to settle the above mentioned open question and to show that clean rings indeed form a proper subclass of exchange rings.

A ring $R$ is called left quasi-duo if every maximal left ideal of $R$ is two sided ideal of $R$. Nicholson had proved that commutative exchange rings are clean. Yu in [31] extended Nicholson's result to a larger class of rings, namely the class of left quasi-duo rings.

**Theorem 1.2** (Yu [31]) *For a left or right quasi-duo ring, the following are equivalent:*

*(i) $R$ is an exchange ring.*
*(ii) $R$ is a clean ring.*

A ring $R$ is called von Neumann regular, if each principal right ideal of $R$ is a direct summand of $R$ as a right $R$ module, equivalently if for each $x \in R$, there exists $y \in R$ such that $x = xyx$ and if for each $x \in R$, there exists a unit $u \in R$ such that $x = xux$, then $R$ is called a unit-regular ring. Though Bergman's example tells that not all von Neumann regular rings are clean but Camillo and Khurana in [6] proved that all unit-regular rings are clean.

**Theorem 1.3** (Camillo, Khurana [6]) *A ring $R$ is unit regular if and only if every element $a \in R$ can be written as $e + u$ such that $aR \cap eR = 0$, where $e$ is an idempotent and $u$ is a unit in $R$.*

It is also easy to see that a clean ring may not be unit-regular or regular. For example, $\mathbb{Z}_4$ , that is the ring of integers modulo $4$ is a clean but it is not von Neumann regular. The following result by Lee et al. characterizes von Neumann regular rings that are clean:

**Theorem 1.4** (Lee et al. [18]) *A regular ring is clean if and only if its indecomposable images are clean.*

The following characterization of semiperfect rings in terms of clean rings is due to Camillo and Yu [8]:

**Theorem 1.5** (Camillo, Yu [8]) *A ring $R$ is semiperfect if and only if $R$ is clean and contains no infinite set of orthogonal idempotents.*

If $V$ is a finite dimensional vector space over a division ring $D$, the End $(V_{D)}$ is isomorphic to $M_n(D)$ for some $n$, which can be seen to be a clean ring in many ways. For instance, it is a unit-regular ring and hence clean. Another way to see this is that $M_n(D)$ is simple artinian ring and hence is a semiperfect ring and is therefore clean. Nicholson and Varadarajan [22] proved that the result is also true for any countable dimensional vector space over a division ring.

**Theorem 1.6** (Nicholson, Varadarajan [22]) *If $V_D$ is a vector space of countably infinite dimension over a division ring $D$, then End $(V_{D)}$ is clean.*

Nicholson and Varadarajan asked whether their result is true for vector spaces of arbitrary dimension also. For vector spaces of arbitrary dimensions, result was independently proved by Searcóid [26].

**Theorem 1.7** (Searcóid [26] ) Let $X$ be a linear space over a field $F$, and $T \in L(X)$, the algebra of linear operators on $X$. Then there exists $P \in L(X)$, with $P = P^2$ such that $T - P$ is invertible in $L(X)$.

Although, Searcóid proved his result for linear spaces over division rings, but Nicholson et al. in [24] modified his argument a bit for it to work over division rings too.

A subset $X$ of a ring $R$ is called *left* $T$*- nilpotent* if, for any sequence of elements $\{x_1, x_2, x_3, \ldots\} \subseteq X$, there exists a positive integer $n$ such that $x_1 x_2 \ldots x_n = 0$. A ring $R$ is called *left-perfect* if, $R/J(R)$ is semisimple and $J(R)$ is left $T$- nilpotent.

Besides modifying Theorem 1.6 for vector spaces over division rings, Nicholson et al. [24] also extended the result to the projective modules over left- perfect rings as following:

**Theorem 1.8** (Nicholson [24]) *For any projective $R$- module $P_R$ over a right perfect ring $R$, the endomorphism ring $End\ (P_R)$ is clean.*

Earlier, it was proved by Han and Nicholson [16] that if $R$ is a clean ring, then any finite matrix ring $M_n(R)$ is also clean, in other words, endomorphism ring of any finitely generated free module over a clean ring is also clean. But the following result due to Camillo et al. [7] tells that if $R$ is a clean ring, then endomorphism ring of a free module over $R$ may not be clean. In particular, the following result tells that if $R$ is a semilocal ring, that is not right perfect, then endomorphism ring of the free module $R^{(\mathbb{N})}$ is not clean.

**Theorem 1.9** (Camillo et al. [7]) *Let $R$ be a ring and $F_R = R^{(\mathbb{N})}$. The following are equivalent:*

*(1) For every right projective module $P_R, End\ (P_R)$ is clean and $End\ (P_R)/J(End(P_R))$ is regular.*
*(2) $End(F_R)$ is clean and $End(F_R)/J(End(F_R))$ is regular.*
*(3) $End(F_R)$ is clean and $R$ is semilocal.*
*(4) $R$ is right perfect.*

A module $M$ is called *continuous* if it satisfies the following two properties:

(a) Every submodule of $M$ is essential inside a direct summand of $M$.
(b) Every submodule of $M$ that is isomorphic to a summand of $M$ is itself a summand of $M$.

It was proved by Mohamed and Müller [20] that if $M$ is a continuous module, then End $(M)$ has the finite exchange property, which in view of Warfield's result in [29] implies that End$(M)$ is an exchange ring. Camillo et al .[7] proved that End$(M)$ is infact a clean ring in this case.

**Theorem 1.10** (Camillo et al. [7]) *If $M$ is a continuous module, then End $(M)$ is a clean ring.*

### 2. Strongly Clean Rings

An element $a$ in a ring $R$ is called *strongly clean* if there exits an idempotent $e$ and unit $u$ in $R$ with $eu = ue$ such that $a = e + u$ and the ring $R$ is called *strongly clean* if each element of $R$ is strongly clean. It is easy to see that idempotents, units, nilpotent elements and all elements of the Jacobson radical of the ring are strongly clean. So it is clear that every local ring is strongly clean. Strongly clean rings were first introduced and studied by Nicholson in [23]. Strongly clean rings have deep connections with *strongly* $\pi$-regular *rings.*

An element $a \in R$ is called *right $\pi$- regular* if the chain $aR \supseteq a^2R \supseteq a^3R \supseteq \cdots$ terminates, that is, there exists $n \geq 1$ such that $a^nR = a^{n+1}R$ and a ring is called *right $\pi$- regular ring* if each element of $R$ is right $\pi$ – regular. Left $\pi$- regular rings are defined similarly. For many years the left and right versions of these concepts were treated differently before Dischinger proved this remarkable result:

**Theorem 2.1** (Dischinger [14]) *Every right $\pi$- regular ring is left $\pi$-regular also*.

But a single right $\pi$-regular element may not be left $\pi$-regular. For instance, if $R$ is a ring which is not directly finite, then there exists $a, b \in R$ such that $ab = 1$ but $ba \neq 1$, then $aR = a^2R = R$, implying that $a$ is right $\pi$-regular but the chain $Ra \supseteq Ra^2 \supseteq Ra^3 \supseteq \cdots$ doest not terminate, as otherwise $a$ will be left invertible.

An element which is both left and right $\pi$ - regular is called *strongly $\pi$-regular* and a ring $R$ is called *strongly $\pi$-regular ring* if each element of $R$ is strongly $\pi$-regular. For example, every algebraic algebra is strongly $\pi$-regular. Also, one sided perfect ring is strongly $\pi$-regular, as a ring $R$ is left (resp. right) perfect if and only if $R$ satisfies descending chain condition on principal right (resp. left) ideals. The following result was first proved by Burgess and Menal [4] using sheaf theoretic arguments:

**Theorem 2.2** (Burgess, Menal [4]) *In a strongly $\pi$- regular ring, every element can be uniquely written as sum of an idempotent and a unit that commute with each other.*

Later, the following element wise result with an elementary proof was given by Nicholson [23]:

**Theorem 2.3** (Nicholson [23]) *If $a \in R$ is strongly $\pi$- regular, then $a$ is strongly clean.*

Converse of Theorem 2.2 is not true. For instance, if $R = \mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at the ideal $2\mathbb{Z}$, then $R$ is a local ring and hence it is strongly clean but $a = \frac{2}{1}$ is not strongly $\pi$-regular, as if $a^m = a^k\, r$ for some $r \in R$, and positive integers $m \geq k$, then $a^m(1 - a^{k-m}\, r) = 0.$, which is not possible as $R$ is an integral domain and $a$ is neither a unit and nor zero. The following characterization of strongly clean elements in the endomorphism ring of a module is given by Nicholson [23]:

**Theorem 2.4** (Nicholson [23]) *Let $E = End\ ({}_RM)$*, the following are equivalent for $\alpha \in E$:

1. *$\alpha$ is strongly clean in $E$.*
2. *There exists $\pi^2 = \pi \in E$ such that $\alpha\pi = \pi\alpha$, $\alpha\pi$ is a unit in $\pi E \pi$ and $(1 - \alpha)(1 - \pi)$ is a unit in $(1 - \pi)E(1 - \pi)$.*
3. *$M = P \oplus Q$, where $P$ and $Q$ are $\alpha$- invariant and $\alpha_{|P}$ and $(1 - \alpha)_{|Q}$ are isomorphisms.*
4. *$M = P \oplus Q$, where $P$ and $Q$ are $\alpha$- invariant, $Ker\ \alpha \subseteq Q \subseteq M(1 - \alpha)$ and $Ker\ (1 - \alpha) \subseteq P \subseteq M\alpha$.*
5. *$M = P_1 \oplus P_2 \oplus \ldots \oplus P_n$ for some $n \geq 1$, where $P_i$ is $\alpha$-invariant and $\alpha_{|P_i}$ is strongly clean in $End\ (P_i)$ for each $i$.*

It is easy to see that direct product of rings is strongly clean if and only if each of the factors is strongly clean and every homomorphic image of a strongly clean ring is also strongly clean. However the converse is not true. Borooah [2] gave an example of a ring R with a ring endomorphism $\sigma$ such that the triangular matrix ring $T_2(R)$ is strongly clean but $T_2(R[[x;\sigma]])$ is not strongly clean. Diesl et al. [12] proved the following result in this direction.

**Theorem 2.5** (Diesl et al. [12]) *Let R be a ring complete with respect to an ideal $I$ (which is necessarily contained in the Jacobson radical of $R$). Let $x \in R$, and let $\bar{x}$ denote the image of $x$ in $\bar{R} = R/I$. If $\bar{x}$ is strongly $\pi$-regular in $R/I$, then $x$ is strongly $\pi$-clean with respect to $I$. In particular, $x$ is strongly clean in $R$. If, additionally, the image of $x$ in $R/I$ is uniquely strongly clean, then $x$ is uniquely strongly clean.*

The following result was also proved by Nicholson [23]:

**Proposition 2.6** (Nicholson [23]) *If $e^2 = e$ and $a \in eRe$ is strongly clean in $eRe$, then $a$ is strongly clean in $R$.*

Nicholson also asked if corner ring of a strongly clean ring is also strongly clean. This question was answered in positive by Chen [9]. Chen also gave a new characterization of strongly clean rings. He called a ring $R$ *strongly exchange ring* if for each $a \in R$, there exists an idempotent $e \in R$ and elements $x, y \in R$ such that $e = ax = xa$ and $1 - e = (1-a)y = y(1-a)$. Chen proved the following:

**Theorem 2.7** (Chen [9]) *A ring $R$ is strongly clean if and only if $R$ is strongly exchange.*

Though, Chen proved that corner ring of a strongly clean ring is also strongly clean but he showed that a single strongly clean element may not behave that way, that is, he showed that if $a \in R$ is strongly clean and $e$ is an idempotent in $R$, then $eae$ may not be strongly clean in $eRe$.

As mentioned earlier, every semiperfect ring is clean, Nicholson in [23] asked whether every semiperfect ring is strongly clean also. This question was answered in negative by Wang and Chen [30]. They showed that the ring $M_2(\mathbb{Z}_{(2)})$ is semiperfect but it is not strongly clean. Also, as $\mathbb{Z}_{(2)}$ is a local ring, it is strongly clean but $M_2(\mathbb{Z}_{(2)})$ is not strongly clean, implying that the property of being strongly clean is not Morita invariant.

Chen et al. [10] studied those commutative local rings for which $2 \times 2$ matrix rings over them are strongly clean. They proved the following:

**Theorem 2.8** (Chen et al. [10]) *Let $R$ be a commutative local ring.*

*(1) If $2 \in U(R)$, then $M_2(R)$ is strongly clean iff for all $w \in J(R)$, $x^2 - x = w$ is solvable in $R$.*

*(2) If $\frac{R}{J(R)} \cong \mathbb{Z}_2$, then $M_2(R)$ is strongly clean iff for all $w \in J(R)$, $x^2 - x = w$ is solvable in $R$.*

*(3) If $2 \in J(R)$, and $\frac{R}{J(R)} \ncong \mathbb{Z}_2$, then $M_2(R)$ is strongly clean iff for all $w_1, w_2 \in J(R)$, $x^2 + (1+w_1)x = w_2$ is solvable in $R$.*

Borooah et al. [3] proved the following:

**Theorem 2.9** (Borooah et al. [3]) *If $R$ is a commutative local ring, $n \geq 2$ and $h \in R[x]$ is a polynomial of degree $n$, then the following are equivalent:*

1. *Any $A \in M_n(R)$ such that characteristic polynomial of $A$ is equal to h, is strongly clean.*
2. *The companion matrix of $h$ is strongly clean in $M_n(R)$.*

## 3. Uniquely Clean Rings

Anderson and Camillo [1] called a ring $R$ *uniquely clean ring* if every element of $R$ can be uniquely written as sum of a unit and idempotent. Anderson and Camillo studied the uniquely clean rings in case of commutative rings and later Nicholson and Zhou [25] generalized most of their results on commutative uniquely clean rings to the non-commutative case. They observed that every idempotent in a uniquely clean ring is central and that direct product of rings is uniquely clean if and only if each of the factors is uniquely clean, as a result, it follows that corner ring of a uniquely clean ring is uniquely clean. They also observed that uniquely clean rings have to be directly finite and every uniquely clean ring is left and right quasi-duo. They gave the following characterization of local uniquely clean rings:

**Theorem 3.1** (Nicholson, Zhou [25]) *The following are equivalent for a ring $R \neq 0$:*

(1) *$R$ is local and uniquely clean.*

(2) *$R$ is uniquely clean and the only idempotents in $R$ are $0$ and $1$.*

(3) $R/J \cong \mathbb{Z}_2$.

Further, they observed that if $R$ is a uniquely clean ring, then $R/J$ has characteristic $2$. The following characterization of uniquely clean rings is also due to Nicholson and Zhou [23]:

**Theorem 3.2** (Nicholson, Zhou [25]) *The following are equivalent for a ring $R$:*

*(1) $R$ is uniquely clean.*

*(2) $R/J$ is Boolean and idempotents lift uniquely modulo $J$.*

*(3) $R/J$ is Boolean, idempotents lift modulo $J$ and idempotents in $R$ are central.*

*(4) For each $a \in R$, there exists a unique idempotent $e \in R$, such that $e - a \in J$.*

Chen in [11] proved the following:

**Theorem 3.3** (Chen [11]) *A ring $R$ is uniquely clean if and only if:*

*(1) $R$ is an exchange ring with all idempotents central.*

*(2) For all maximal ideals $M$ of $R$, $R/M \cong \mathbb{Z}_2$.*

A ring is said to be of stable range one if $aR + bR = R$, for any $a, b \in R$, implies that $a + br \in U(R)$ for some $r \in$ R. As already discussed, unit regular rings are clean and regular

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rings with stable range 1 are unit-regular, it would be interesting to know if arbitrary exchange rings with stable range 1 are also clean.

**References**


1. Anderson, D.D., Camillo, V. P., (2002): 'Commutative rings whose elements are a sum of a unit and idempotent'. *Communications in Algebra, 30(7),* 3327-3336.
2. Borooah, G., Diesl, A. J., (2007): 'Strongly clean triangular matrix rings over local rings'. *J. Algebra 312 (2),* 773-797.
3. Borooah, G., Diesl, A. J., (2008): 'Strongly clean matrix rings over commutative local rings'. *J. Pure Appl. Algebra 212 (1),* 281-296.
4. Burgess, W. D., Menal, P., (1988): 'On strongly $\pi$-regular rings and homomorphisms into them'. *Communications in Algebra, 16,* 1701-1725.
5. Crawley, P., Jonsson, B.,(1964): 'Refinements for infinite direct decompositions of algebraic systems.' *Pacific J. Math, 14, 797-855.*
6. Camillo, V. P. Khurana, D., (2001): 'A Characterization of unit-regular rings'. *Communications in Algebra, 29(5),* 2293-2295.
7. Camillo, V. P. et al., (2006): 'Continuous modules are clean'. *Journal of Algebra, 304,* 94-111.
8. Camillo, V.P., Yu, H.P., (1994): 'Exchange rings, units and idempotents'. *Communications in Algebra, 22(12),*4737-4749.
9. Chen, W., (2006): 'A question on strongly clean rings'. *Communications in Algbera, 34,* 2347-2350.
10. Chen J. et al., (2006): 'When is the $2 \times 2$ matrix ring over a commutative local ring strongly clean ?' *Journal of Algebra, 301,* 280-293.
11. Chen, H. , (2011): 'On Uniquely Clean Rings', *Communications in Algebra, 39,* 189-198.
12. Diesl, A. J. et al., (2014):'A Note on Completeness and Strongly Clean Rings'. *Journal of Pure and Applied Algebra, 218, 661-665.*
13. Contessa, M. (1982): 'On *pm*-rings'. *Communications in Algebra, 10(1),* 93-108.
14. Dischinger, M. F. (1976): 'Sur les anneaux fortement vr-reguliers'. *C. R. Aca. Sc. Paris, 283,* 571-573.
15. Engelking, R., (1989): General topology, Sigma series in Pure Mathematics, Vol. 6, Heldermann, Berlin.
16. Han, J. , Nicholson, W. K., (2001): 'Extensions of clean rings', *Comm. Algebra, 29,* 2589-2595.
17. Handelman, D., (1977): 'Perspectivity and cancellation in regular rings'. *Journal of Algebra, 48,* 1-16.
18. Lee, T. K. et al. (2008): 'An example of Bergman's and the extension problem for clean rings'. *Communications in Algbera, 36,* 1413-1418.
19. McGovern, W. , (2006): 'Neat Rings', *Journal of Pure and Applied Algebra, 205(2),* 243-265.
20. Mohamed , S. H. , Müller, B. J., (1989) 'Continuous modules have the exchange property', Abelian Group Theory, Contemp. Math. Vol. 87, Amer. Math. Soc.,Providence, RI, 285-289.
21. Nicholson, W.K., (1977): 'Lifting idempotents and exchange rings.' *Trans. Amer. Math. Soc., 229,* 269-278.
22. Nicholson, W. K., Varadarajan, K. (1998): 'Countable linear transformations are clean'. *Proc. Amer. Math. Soc. 126,* 61-64.
23. Nicholson, W. K., (1999): 'Strongly clean rings and Fitting's Lemma'. *Communications in Algebra, 27(8),* 3583-3592.
24. Nicholson, W. K., Varadarajan, K. and Zhou. Y. (2004): 'Clean endomorphism rings'. *Arch. Math. (Basel), 83,* 340-343.
25. Nicholson, W.K., Zhou, Y.,(2004): 'Rings in which elements are uniquely the sum of an idempotent and a unit'. *Glasgow. Math. J., 46,* 227-236.
26. Searcóid, M. Ó. (1997): 'Perturbation of linear operators by idempotents'. *Irish Math. Soc. Bull. 39,* 10-13.
27. Šter, J., (2012): 'Corner rings of a clean ring need not be clean'. *Comm. Algebra, 40,* 1595-1604.
28. Vasconcelos, W. V., (1973): 'Finiteness in projective ideals'. *Journal of Algebra, 25,* 269-278.
29. Warfield, R. B. (1972): 'Exchange rings and decompositions of modules.' *Math. Ann. 199,* 31-36.
30. Wang, Z. , Chen. J., (2004): 'On two open problems about strongly clean rings'. *Bull. Austr. Math. Soc., 70,* 279-282.
31. Yu, H. P., (1995): 'On quasi-duo rings'. *Glasgow Mathematics Journal, 37(01),* 21-31.