RESEARCH REVIEW JOURNALS
Volume-04
Issue-06
June-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

Numerical Analysis and Optimal Control of PDE Theory

[1]Ritesh & [2]Dr. Dhanpal Singh

[2]*Assistant Professor (Mathematics), Keshav Mahavidyalaya (University of Delhi), New Delhi*

ARTICLE DETAILS

Article History
Published Online: 12 June 2019

Keywords
PDE theory, Control Theory.

ABSTRACT

Differential equations are extremely useful tools in modelling all sorts of dynamical systems. As mathematicians, reading them for the wellbeing of their own is an altogether adequate, and even praiseworthy endeavor.
Shockingly, numerous individuals and organizations that supply real award cash have various sensibilities, and need 'results' that are 'helpful' and 'commonsense'. Most human undertakings include some level of attempting to control the conduct of some framework or another. That is, a typical subject in human idea is, "given that I know how this framework will carry on, how might I change it, to make the framework demonstration such that I would like?"
The primary objective of Control Theory is to respond to this first address for dynamical frameworks demonstrated by differential conditions (the last two are left for guard dog associations and history specialists, individually).

1. Introduction

In a control problem we find the following basic elements.

(1) A control u that we can handle according to our interests, which can be picked among a group of practical controls K.

(2) The condition of the framework y to be controlled, which relies upon the control. A few confinements can be forced on the state, in scientific terms y C, which implies that only one out of every odd conceivable condition of the framework is palatable.

(3) A state condition that builds up the reliance between the control and the state. In the following segments this state condition will be a halfway differential condition, y being the arrangement of the condition and u a capacity emerging in the condition with the goal that any adjustment in the control u delivers an adjustment in the arrangement y. Anyway the inception of control hypothesis was associated with the control of frameworks represented by conventional differential conditions and there is an immense movement in this field.

(4) A capacity to be limited, called the target work or the cost capacity, contingent upon the control and the state (y, u).

The goal is to decide an acceptable control, called ideal control, that gives a palatable state to us and that limits the estimation of utilitarian J. The fundamental inquiries to think about are the presence of arrangement and its calculation. Anyway to acquire the arrangement we should utilize some numerical techniques, emerging some fragile scientific inquiries in this numerical investigation. The initial step to tackle numerically the issue requires the discretization of the control issue, which is made ordinarily by limited components. A characteristic inquiry is the manner by which great the guess is, obviously we might want to have some mistake assessments of these approximations. So as to infer the mistake gauges it is fundamental to have some normality of the ideal control, some request for differentiability is important, probably a few subordinates in a powerless sense. The normality of the ideal control can be found from the principal request optimality conditions. Another key apparatus in the verification of the mistake gauges is the utilization of the second request optimality conditions. In this way our investigation requires to determine the first and second request conditions for optimality.

When we have a discrete control issue we need to utilize some numerical calculation of streamlining to take care of this issue. At the point when the issue isn't curved, the improvement calculations normally gives nearby minima, the inquiry presently is if these neighborhood minima are critical for the first control issue.

The accompanying advances must be pursued when we study an ideal control issue:

(1) Existence of a solution.
(2) First and second request optimality conditions.
(3) Numerical estimate.
(4) Numerical goals of the discrete control issue.

We won't talk about the numerical calculations of advancement, we will just think about the initial three points for a model issue. In this model issue the state condition will be a semilinear elliptic fractional differential condition. Through the nonlinearity presents a few difficulties in the investigation, we have liked to consider them to show the pretended constantly request optimality conditions. Without a doubt, if the condition is straight and the cost useful is the normal quadratic utilitarian, at that point the utilization of the second request optimality conditions is covered up.

There are no numerous books dedicated to every one of the inquiries we are going to examine here. Right off the bat let me notice the book by Profesor J.L. Lions [38], which is an obliged reference in the investigation of the hypothesis of ideal control issues of halfway differential conditions. In this content, has left a permanent track, the peruser will have the option to discover a portion of the techniques utilized in the goals of the two first inquiries above demonstrated. Later books are X. Li and J. Yong [37], H.O. Fattorini [34] and F. Tr¨oltzsch [46].

Control versus Optimal Control

The astute reader may have noticed that 'Optimal Control' is comprised of two words, and that the first is a modifier of the second. There is a reason for this; Control Theory and Optimal Control Theory ask two different, but related, questions. Let's investigate.

An exceptionally (perhaps lamentably) general version of the optimal control problem is as follows: given some sort of dynamical system, together with some way to control it, which may be modelled as

$$\Lambda y = Bu$$

Where Λ and B are operators, specifying the model of the system, and how the control acts on the system, respectively, y is the state of the system, and u is the control, from some admissible set of controls, U.

We are additionally given some 'cost functional' J(u), which maps to the reals, express- ing the 'cost of running the control u'. The question Optimal Control theory asks is to find a control u, in the space of admissible controls, such that

$$J(u) = inf_{u\in U} J(u)$$

The slight peccadillo here is the following: how ought we choose the set of admissible con- trols? What would we want out of these controls in the first place? Obviously, it would do us no good if we found some control that minimized the cost functional, but this control didn't produce the desired results.

There are many types of control problems, depending on the context of the system un- der study, and the objectives of the controllers. Three types are generally considered, it is convenient to introduce some notation here:

Deftnition 1. *Let U be a function space and let y_1 be some given initial data in some Hilbert space V, let T>0, then we define the set of reachable states in time T*

$$R(T ; y_1) = \{y(T) : \text{y is a solution of } (*), \text{ with } u \in U\}$$

Here, U will be chosen based on the properties desired of our control. Control theory generally considers three types of controllability:

Deftnition 2. *We say system (*) is exactly controllable in time T if for every initial data y_1, $R(T;y_1)=V$.*

Deftnition 3. *System (*) is said to be approximately controllable in time T if for every initial data y_1, $R(T;y_1)$ is dense in V.*

Deftnition 4. *System (*) is said to be null controllable in time T if, for every initial data y_1, we have* $0 \in R(T ; y_1)$

Depending on the type of controllability required by the application, the Optimal Control setting will take as its set of admissible controls the control space that permits the type of controllability needed by the application. The question of control theory is then "for what types of controls does this system have the desired controllability properties?"

Finite Linear Systems

In order to introduce some of the concerns of control theory in a more concrete setting, let's develop the theory of controllability for finite, linear systems, which turns out to have an exceptionally elegant answer. In this section, we consider the system:

$$y^j = Ay + Bu \text{ , } t \in (0, T) \qquad (1)$$

$$y_0 = y(0) \in R^n$$

We have A ∈ M(n,n), B ∈ M(n,m), m≤n.

The controllability problem for this class of systems has a satisfying algebraic answer. Our program for this section will be as follows; first, we will show the equivalence of all three types of controllability for finite linear systems, then we'll prove Kalman's rank condition, which characterizes exactly when a system is controllable by the rank of a certain matrix.

First, let's establish the link between exact and null controllability:

Note that linearity of this system immediately gives us an equivalence between null control- lability and exact controllability, since, for T>0, if y(T)=y_T f= 0 then we can define: z=y-x solving the system:

$$z^j = Az + Bu$$

$$z(0) = y(0) - x(0)$$

Where x is a solution to the system:

$$x^j = Ax$$

$$x(T) = y_T$$

Then y(T)=y_T iff z(T)=0. So due to the linear, finite nature of the system, we have that null controllability and exact controllability are equivalent. When we examine the heat equation, we'll see that this is not the case in an infinite dimensional context.

In a similarly quick way, we can brush concerns of approximate controllability under the rug by noting that the set of reachable states is affine (explicitly writing y(T) out as a func- tion of y(0) and u with the variation of constants formula makes this clear), and for finite n, the only dense affine subspace of R^n is R^n, so in the finite case, approximate controllability is equivalent to exact controllability.

Let's concern ourselves now with characterizing exactly which systems are controllable (in any of these three senses).

It is often convenient to examine the properties a system that is strongly related to our prime system, that is the adjoint system of (1). Letting A^* be the adjoint of A, then we consider the adjoint system, which runs backward in time:

$$-\dot{\varphi} = A^*\varphi, \quad t \in (0, T) \quad (2)$$

$$\varphi(T) = \varphi_T$$

This system gives rise to the dual notion to controllability, known as observability. Deftnition 5. *The adjoint system is said to be observable in time T>0 if ∃c>0 such that*

$$\int_0^T |B^*\varphi|^2\, dt \geq c|\varphi(0)|^2$$

For all $\varphi_T \in \mathbb{R}^n$, *with φ the corresponding solution to the adjoint system.*

The above inequality is often referred to as the observation inequality. The concept of observability makes concrete the general notion that "the action of the controls is sufficient to determine the state of the system". In particular, the observation inequality guarantees that the solution of the adjoint equation at t=0 is completely determined by the $B^*\varphi$ term, which is the quantity observed through the control.

Theorem 1. *Let H be a reflexive Banach space, K a closed convex subset of H and φ : K → R*

a function such that:

φ is convex

φ is lower semi-continuous

φ If K in unbounded, then φ is coercive Then φ attains *its minimum in K.*

Now, we're in the position to establish the correspondence between observability and opti- mality:

Theorem 2. *System (1) is exactly controllable in time T iff the adjoint system is observable in time T.*

Proof. (Observability ⇒ controllability)

By the above lemma, if the adjoint system is observable in time T, then it suffices to show that $\forall y_0 \in \mathbb{R}^n$, the functional J has a minimum.

By the DMCV, since J is continuous, it suffices to show that J is coercive, but the observation inequality gives us that

$$\int_0^T |B^*\varphi|^2\, dt \geq c|\varphi(0)|^2 \quad \forall \varphi_T \in \mathbb{R}^n$$

So that, certainly,

$$J(\varphi_T) > \frac{c}{2}|\varphi_T|^2 - |(y_0, \varphi(0))|$$

Which gives us coercivity.

(Controllability ⇒ Observability)

Theorem 3. *System (1) is exactly controllable for some time T>0 iff*

rank$[B, AB, ..., A^{n-1}B] = n$

Moreover, if the system is controllable for some time T>0, then it is controllable for all time.

Proof. (⇒) Suppose that $rank[B, AB, ..., A^{n-1}B] < n$, then the rows are linearly dependent, so there exists some $v \in \mathbb{R}^n$ with $vf=0$ and

$v^T[B, AB, ..., A^{n-1}B] = [v^T B, v^T AB, ..., v^T A^{n-1}B] = 0$

Which gives that $v^T A^k B = 0$, for k={0, 1, ..., $n-1$}

By the Cayley-Hamilton theorem, we have the existence of a polynomial q, such that q(A)=0 and thus, there exist constants $c_1, ..., c_n$ such that

$A^n = c_1A^{n-1} + ... + c_nI$

but by the above, multiplying by v^T gives $v^T A = 0$. So that, in fact, we have $v^T A^kB = 0$, $\forall k \in \mathbb{N}$.

This gives us that $v^T e^{At}B = 0$, ∀t,

but the variation of constants formula tells us that the state of the system at any point is given by

$$y(t) = e^{At}y_0 + \int_0^t e^{A(t-s)} Bu(s)\, ds$$

Taking the inner product against v kills the integrand, giving:

$(v, y(T)) = (v, e^{aT} y_0)$

So the projection of the solution along v is independent of the control, hence the system is uncontrollable.

(⇐) Suppose now that $rank([B, AB, ..., A^{n-1}B]) = n$, we know that it suffices to show that the system is observable. For which it suffices to show that $B^*\varphi(t) = 0, \forall t \in [0, T] \Rightarrow \varphi_T = 0$.

So, suppose $B^*\varphi = 0$.

Since $\varphi(t) = e^{A^*(T-t)}\varphi_T$, we have that $0 = B^*e^{A(T-t)}\varphi_T$, for all 0 t T . Taking derivatives in T yields that $B^*[A^*]\varphi_T = 0$ for all k>0.

This implies that $[B^*, B^*A^*, ..., B^*(A^*)^{n-1}]\varphi_T = 0$, but since $[B, AB, ..., A^{n-1}B]$ is of full rank, so is $[B^*, B^*A^*, ..., B^*(A^*)^{n-1}]$, hence $\varphi_T = 0$. Which is our desired result.

Controllability of the Heat Equation

As an example of how one extends the techniques and concerns of the finite linear case to the infinite case, where exact, approximate and null controllability are no longer equivalent, let us consider now the heat equation:

Let $\Omega \subset \mathbb{R}^n$ and $\Gamma = \partial\Omega$, T>0 we consider:

$y_t - Oy = u$ in $\Omega \times (0, T)$

$y = 0$ *on* $\Gamma \times (0, T)$

$y(x, 0) = y_0(x)$ *in* Ω

With $\text{supp}(u) := \omega \subset \Omega$. This is known (for obvious reasons) as the *interior control problem*. We consider here the questions of approximate and exact controllability, as the question of null controllability, while answered in the affirmative, is considerably more involved.

Optimal Control and Hamilton-Jacobi Equations

We would like to approach the problem of constructing optimal controls. To that end, consider the following problem:

$$\dot{x}(s) = \boldsymbol{f}(x(s), \alpha(s)) \quad (t < s < T)$$

$$x(t) = x \in \mathbb{R}^n$$

With α a control, and we take the space of admissible controls, U, to be the set of measurable functions on [0,T], with T>0 a fixed time. We define the cost functional to be:

$$C_{x,t}(\alpha) = \int_t^T L(x(s), \alpha(s))\,\mathrm{d}s + \psi(x(T))$$

$$C_{x,t}(\alpha) = \quad_{t} \; L(x(s), \alpha(s))\,\mathrm{d}s + \psi(x(T))$$

And our goal is to find some α that minimizes this functional. In our current setting, we suppose that f, L and ψ are all bounded and Lipschitz continuous on their domain of definition. As it turns out, there is a useful result that gives us a starting place for this endeavour, giving necessary conditions for such a control to be optimal.

2. Conclusion

Optimal control problems, particularly for partial differential equations, are often ill-posed and need to be regularized to obtain good approximations. We here utilize the hypothesis of the relating Hamilton-Jacobi-Bellman conditions to develop regularizations and determine mistake gauges for ideal plan issues. The developed Pontryagin strategy is a straightforward and general technique where the main, systematic, advance is to regularize the Hamiltonian. Next its Hamiltonian framework is figured productively with the Newton strategy utilizing a meager Jacobian. A blunder gauge for the contrast among correct and inexact target capacities is inferred, depending just on the distinction of the Hamiltonian and its limited dimensional regularization along the arrangement way and its L2 projection, for example not on the distinction of the correct and surmised answers for the Hamiltonian frameworks.

References

1. Lawrence C. Evans, Partial Differential Equations: Second Edition. AMS 1998.
2. Jacques Louis Lions, Optimal Control of Systems Governed by Partial Differential Equa- tions. AMS 1970.
3. Sorin Micu, Enrique Zuazua, An Introduction to the Controllability of Partial Differential Equations. www.univ-orleans.fr/mapmo/CPDEA/Cours/Zuazua/micu-zuazua.pdf. Ac- cessed on Fri. Nov 23, 2012.
4. Jerzy Zabczyk, Mathematical Control Theory: An Introduction. Birkhäuser 1995.
5. Alberto Bressan, Viscosity Solutions of Hamilton-Jacobi Equations and Optimal Control Problems. www.math.psu.edu/bressan/PSPDF/hj.pdf. Accessed on Sat. Dec 1, 2012.
6. Heinz W. Engl, Martin Hanke, and Andreas Neubauer. Regularization of inverse problems, volume 375 of Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1996.
7. FEniCS. FEniCS project. URL: urlhttp//www.fenics.org/.
8. Ernst Hairer, Christian Lubich, and Gerhard Wanner. Geometric numerical integration, volume 31 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, second edition, 2006. Structure-preserving algorithms for ordinary differential equations.
9. J. Hoffman, J. Jansson, A. Logg, and G. N. Wells. DOLFIN. URL: urlhttp//www.fenics.org/dolfin/.
10. J.-L. Lions. Optimal control of systems governed by partial differential equations. Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York, 1971.
11. M. Sandberg and A. Szepessy. Convergence rates of symplectic Pontryagin approximations in optimal control theory. M2AN, 40(1), 2006.
12. Mattias Sandberg. Convergence rates for numerical approximations of an optimally controlled Ginzburg-Landau equation. preprint, 2006.
13. Curtis R. Vogel. Computational methods for inverse problems, volume 23 of Frontiers in Applied Mathematics.