RESEARCH REVIEW JOURNALS
Volume-04
Issue-06
June-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

Some Investigations on Exact Solutions of Einstein's Field Equations with Specified Equation of State

[1]Sunita Singh & [2]Dr. Manish Kumar

[1]*Research Scholar, University Department of Mathematics, Magadh University, Bodh-Gaya (India)*
[2]*Associate Professor, Department of Mathematics, A.N. College, Patna (India)*

ARTICLE DETAILS

Article History
Published Online: 12 June 2019

Keywords
Exact solution, cosmology, equation of state, fluid sphere, shear tensor, metric space.

ABSTRACT

In this paper we have obtained some exact static spherical solution of Einstein's field equation with cosmological constant $\wedge = 0$ and equation of state $p = \alpha\rho$ (taking suitable choice of g_{11} and g_{44}). We have taken $e^{\beta} = lr^{5/4}$ and $e^{-\alpha} = c$ (where l is a constant), which help to investigate the value of e^{α} and e^{β} respectively. Many previously known solutions are contained herein as a particular case. Various physical and geometrical properties have been studied.

1. Introduction

Various researcher in theory of relativity have focused their mind to the study of solution of Einstein's field equation with cosmological constant $\wedge = 0$ and equation of state $p = \rho$. Solution of Einstein's field equation with equation of state $p = \rho$ have been obtained by various authors e.g., Latelier [7], Letelier and Tabensky [8], Tabensky and Taub [13] and Yadav et al. [19]. Singh and Yadav [11] have also discussed the static fluid sphere with the equation of the state $p = \rho$. Further study in the line has been done by Yadav and Saini [17], which is more general than one due to Singh and Yadav [11]. Schwarzschild [9] considered the perfect fluid spheres with homogeneous density and isotropic pressure in general relativity and obtained the solutions of relativistic field equations. Tolman [15] developed a mathematical method for solving Einstein's field equations applied to static fluid spheres in such a manner as to provide explicit solutions in terms of known analytic functions. A number of new solutions were thus obtained and the properties of three of them were examined in detail.

Solutions to Einstein's equations with a simple equations of state have been found in various cases, e.g. for $\rho + 3p$ = constant (Whittaker [16]) for $\rho = 3p$ (Klein [5]); for $p = \rho +$ constant (Buchdahl and Land [3], Allunt [1]) and for $\rho = (1 + a)\sqrt{p} + ap$ (Buchdahl [2]). But if one takes, e.g. polytrophic fluid sphere $\rho = ap^{1+\frac{1}{n}}$ (Klein [6]) or a mixture of ideal gas radiation (Suhonen [12]), one soon has to use numerical methods. Davidson [4] has presented a solution a non-stationary analog to the case when $p = \frac{1}{3}\rho$. Tolman [15], Yadav and Purushottam [18], Thomas E Kiess [14], Singh. et.al. [10], and Yadav et al. [20] are some of the authors who have also studied various aspects in this field.

In this paper we have obtained some exact static spherically symmetric solution of Einstein field equation for the static fluid sphere with cosmological constant $\wedge = \vartheta$ and equation of state $p = a\rho$. It has been obtained taking suitable choice of g_{11} and g_{44} (e.g. $e^{\beta} = lr^{5/4}$ and $e^{-\alpha} = c$ (where l is a constant). To overcome the difficulty of infinite density at the centre, it is assumed that distribution has a core of radius r_o and constant density ρ_o which is surrounded by the fluid with the specified equation of state. Various physical and geometrical properties have been also studied.

2. The Field Equations

We consider the static spherically symmetric metric given by

$$ds^2 = e^{\beta} dt^2 - e^{\alpha}dr^2 - r^2d\theta^2 - r^2 \sin^2\theta d\varphi^2 \qquad (2.1)$$

where α and β are functions of r only.

Taking cosmological constant $\wedge$ into account, we obtain the field equations

$$R^i_j - \frac{1}{2} R\,\delta^i_j + \wedge\delta^i_j = -8\pi T^i_j \qquad (2.2a)$$

For $\wedge = 0$, (2.2a) gives

$$R^i_j - \frac{1}{2} R\,\delta^i_j = -8\pi\, T^i_j \qquad (2.2b)$$

For the metric (2.1) are (Tolman [26])

$$-8\pi T^1_1 = e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \qquad (2.3)$$

$$-8\pi T^2_2 = -8\pi T^3_3 = e^{-\alpha}\left(\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta'^2}{4} + \frac{\beta' - \alpha'}{2r}\right) \qquad (2.4)$$

$$-8\pi\, T^4_4 = e^{-\alpha}\left(\frac{\alpha^1}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \qquad (2.5)$$

where a prime denotes differentiation with respect to r.

Though the investigation, we set velocity of light C and gravitational constant G to be unity. A Zeldovich fluid can be regarded as a perfect fluid having the energy momentum tensor.

$$T^i_j = (\rho + p)\, u^i u_j - \delta^i_j\, p \qquad (2.6)$$

Specified by the equation of state

$$p = a\rho \qquad (2.7)$$

We use co-moving co-ordinates so that

$$u^1 = u^2 = u^3 = 0 \text{ and } u^4 = e^{-\frac{\beta}{2}}$$

The non-vanishing components of the energy momentum tensor are

$$T^1_1 = T^2_2 = T^3_3 = -p \text{ and } T^4_4 = \rho$$

We can then write the field equations:-

$$8\pi\rho = e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} \qquad (2.8)$$

$$8\pi\rho = e^{-\alpha}\left(\frac{\beta''}{2} - \frac{\alpha'\beta'}{4} + \frac{\beta''^2}{4} + \frac{\beta' - \alpha'}{2r}\right) \qquad (2.9)$$

$$8\pi\rho = e^{-\alpha}\left(\frac{\alpha'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \qquad (2.10)$$

3. Solution Of The Field Equations

Using equations (2.7), (2.8) & (2.10), we have

$$e^{-\alpha}\left(\frac{\beta'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = e^{-\alpha}\left(\frac{\alpha'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} \qquad (3.1)$$

From [3.1] we see that if β is known, α can be obtained, so we choose –

Case I:

$$e^{\beta} = lr^{5/4} \qquad (3.2)$$

(where l is constant)

Using (3.2) in equation (3.1) goes to form:-

$$\frac{de^{-\alpha}}{dr} + \frac{13}{4r} e^{-\alpha} = \frac{2}{r} \qquad (3.3)$$

Substituting $\tau = e^{-\alpha}$, the equation (3.3) is reduced to,

$$\frac{d\tau}{dr} + \frac{13}{4r}\tau = \frac{2}{r} \qquad (3.4)$$

which is a linear differential equation whose solution is given by:-

$$\tau = \frac{8}{13} + \frac{C}{r^{13/4}} \qquad (3.5)$$

Or

$$e^{-\alpha} = \frac{8}{13} + \frac{C}{r^{13/4}} \qquad (3.6)$$

where C is an integration constant.

Hence the metric (2.1) yields

$$ds^2 = lr^{7/5}\, dt^2 - \left(\frac{10}{17} + \frac{C}{r^{17/5}}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \,.\, d\varphi^2\right)$$

(3.7)

Absorbing the constant l in the co-ordinate differential dt and put C = 0 the metric (3. 7) goes to the form –

$$ds^2 = r^{5/4}\, dt^2 - \frac{13}{8} dr^2 - r^2\left(d\theta^2 + \sin^2\theta \,.\, d\varphi^2\right) \qquad (3.8)$$

The non-zero component of Riemann-christoffel curvature tensor R_{hijk} for the metric (3. 8) is

$$\sin^2\theta\, R_{2424} = R_{3434} = \frac{13}{8} r^{13/4} \sin^2\theta = R_{2323} \qquad (3.9)$$

For the metric (3.8) the fluid velocity v^{i} is given by

$$v^1 = v^2 = v^3 = 0, \qquad v^4 = r^{-5/8} = \frac{1}{r^{5/8}} \qquad (3.10)$$

In the usual notation, we have the rotation $\omega_{ij} = v_{ij} - v_{ji}$, and shear tensor $\sigma_{ij} = \frac{1}{2}\left(v_{ij} + v^{ij}\right) - \frac{1}{3}H_{ij}$ gives results for metric (3.8) as:-

$$\theta = 0,\ \omega_{14} = -\omega_{41} = \frac{-5}{8} r^{-3/8} = \frac{-5}{8r^{3/8}} \qquad (3.11)$$

and

$$\sigma_{14} = \sigma_{41} = \frac{5}{8} r^{-3/8} = \frac{5}{8r^{3/8}}. \qquad (3.12)$$

Case II :

$$e^{-\alpha} = c, \qquad (3.13)$$

where c is constant.

Using (3.13), equation (3.1) goes to the –

$$\beta' + \frac{2}{r}\left[1 - \frac{1}{c}\right] = 0 \qquad (3.14)$$

Now integrate (3.14) w.r.t r we get-

$$e^{\beta} = Ar^{2\left(1-\frac{1}{c}\right)} \qquad (3.15)$$

where A is an integration constant.
If we consider c= 3, then we get -

$$e^{\beta} = Ar^{4/3} \qquad (3.16)$$

Hence, using the equation (3.17) the metric (3.1) yields:-

$$ds^2 = Ar^{4/3}\, dt^2 - 1/3\,(dr^2) - r^2\,(d\theta^2 + \sin^2\theta.\, d\varphi^2) \qquad (3.17)$$

Absorbing the constant A in the co-ordinates differentials dt the metric (3.17) goes to the form :-

$$ds^2 = r^{4/3}\, dt^2 - 1/3\,(dr^2) - r^2\,(d\theta^2 + \sin^2\theta.\, d\varphi^2) \qquad (3.18)$$

Components of curvature tensor, shear, expansion etc. can be calculated as in Case I.

4. Solution For The Perfect Fluid Core

Pressure and density for the metric (3.7-3.8, 3.18) are

$$8\pi p = 8\pi\rho = \frac{9}{4r^2}\left[\frac{8}{13} + \frac{c}{r^{13/4}}\right] - \frac{1}{r^2} \qquad (4.1)$$

If we consider $C = 0$, then equation (4.1) reduces to

$$8\pi p = 8\pi\rho = \frac{9}{4r^2}\left[\frac{8}{13}\right] - \frac{1}{r^2} \qquad (4.2)$$

$$8\pi p = 8\pi\rho = \frac{5}{13r^2} \qquad (4.3)$$

It follows from (4.1-4.3) that the density of the distribution tends to infinity as r tends to zero. In order to get rid of singularity at r=0 in the density we visualize that the distribution has a core of radius r_o and constant ρ_o . The field inside the core is given by Schwarzschild internal solution.

$$e^{-\lambda} = 1 - \frac{r^2}{R^2} \qquad (4.4a)$$

$$e^{\nu} = \left[\bar{L} - \bar{M}\left(1 - \frac{r^2}{R^2}\right)\right]^2 \qquad (4.4b)$$

$$8\pi p = \frac{1}{R^2}\left[\frac{3\bar{M}\left(1 - \frac{r^2}{R^2}\right) - \bar{L}}{\bar{L} - \bar{M}\left(1 - \frac{r^2}{R^2}\right)^{\frac{1}{2}}}\right] \qquad (4.4c)$$

where $\bar{L}$, $\bar{M}$ are constants and $R^2 = \dfrac{3}{8\pi\rho}$.

The continuity condition for the metric (3.7-3.8) and (4.4a-4b-4c) at the boundary gives

$$R^2 = \frac{r_o^2}{\left(\frac{5}{13} - \frac{c}{r_o^{8/13}}\right)} \tag{4.5a}$$

$$\overline{L} = r_o^{5/8} + \frac{5R^2}{8r_o^{11/8}}\left(1 - \frac{r_o^2}{R^2}\right) \tag{4.5b}$$

$$\overline{M} = \frac{5R^2}{8r_o^{11/8}}\left(1 - \frac{r_o^2}{R^2}\right)^{1/2} \tag{4.5c}$$

$$C = r_o^{n+2}\left(\frac{5}{13} - \frac{r_o^2}{R^2}\right) \tag{4.5d}$$

and the density of the core

$$\rho_0 = \frac{3}{8\pi r^2}\left(\frac{5}{13} - \frac{c}{r_o^{8/13}}\right) \tag{4.6}$$

which complete the solution for the perfect fluid core of radius r_o surrounded by considered fluid. The energy condition $T_{ij}\, u^i u_j > 0$ and the Hawking and Penrose condition (Hawking and Penrose, 1970).

$$\left(T_{ij} - \frac{1}{2} g_{ij} T\right) u^i u_j > 0 \, ,$$

Both reduces to $\rho > 0$, which is obviously satisfied.

5. Discussion:

In this chapter we have obtained some exact static spherical solution of Einstein's field equation with cosmological constant $\wedge = 0$ and equation of state $p = a\rho$ and with assumption $e^{\beta} = lr^{5/4}$ and $e^{-\alpha} = c$, which provides several important cases, e.g.- relativistic model, rotation, shear tensor, scalar of expansion.

References

[1] Allnutt, J. A. (1980): Exact Solution of Einstein's Field Equation edited by D. Kramer, H. Stephani, E. Hert, and M. Mac.Callum, Cambridge University Press, Cambridge, 227.
[2] Buchadahl, H. A. (1964): J. Astrophys., 140, 1512.
[3] Buchadahl, H. A. and Land W.J. (1968): J. Australian Math Soc., 8 6.
[4] Davidson, W. (1991) : J. Math. Phys., 32, 1560-61.
[5] Klein, O. (1947): Arkiv. Math Astron Fysik., A34, 1.
[6] Klein, O. (1953): Arkiv. Fysik., 7 ,487.
[7] Letelier, P.S. (1975): J. Math. Phys., 16, 293.
[8] Letelier, P.S. and Tabensky, R.S. (1975): I Nuovo Cimento., 28M, 408.
[9] Schwarzschild, K., Sitz. (1916): Preuss. Akad. Wiss., 189.
[10] Singh, K. P. and Abdussattar (1973): Indian J. Pure Appl. Math., 4, 468.
[11] Singh, T. and Yadav, R.B.S.(1981): J. Math. Phys. Sci., 15, (3) 283.
[12] Suhonen, E. (1968): Kgl. Danske Vidensk Sels. Math. Fys., Medd, 36, 1.
[13] Tabensky, R. and Taub, A.H. (1973): Commun. Math. Phys., 29, 61.
[14] Thomas, E. Kiess. (2009): Class. Quantum grav., A26, 11.
[15] Tolman, R.C. (1939): Phys. Rev., 55, 364.
[16] Whittker, J.M. (1968): Proc. Roy Soc. London A., 306, 1.
[17] Yadav, R.B.S. and Saini, S.L. (1991): Astrophysics and Space Science, 184, 331-336.
[18]. Yadav, R.B.S. and Purushottam. (2001): P.A.S. Jour., 7, 91.
[19] Yadav, R.B.S. et.al. (2007): Acta Ciencia Indica ., Vol. 33, No.-2, 185.
[20] Yadav, A.K., Sharma, A. (2013): Res. Astr. Astrophys., 13, 501.