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Some Common Fixed Points for Mapings with A Contractive Iterate

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ABSTRACT

The present research paper provides mapping which are not necessarily contraction but give fixed point. We have also given an example of continuous function f which satisfies the condition (A) $\left[d(f^{n(x)}(Y)f^{n(x)}(X)\right] \leq kd(y,x)$

where (X, d) is a complete metric space, x ε X, y ε Y but not a contraction.

1. Introduction

Several authors such as Ishikawa [2], Kuhfitting [3] have obtained results concerning common fixed points of finite family of non-expansive mappings by iteration using optional condition. Kuhfitting [3] proved a fixed point theorem with a iteration scheme which converges weakly to a common fixed point of a finite family of non-expansive mappings as a closed convex subset of a uniformly convex Banach space. Reich [4] obtained a result as : Let K be a closed convex subset of a uniformly convex Banach space E with a Frechet differentiable norm T : K → K a non expansive mapping with a fixed points and $\{C_n\}$ a real sequence such that $0 \leq C_n \leq 1$ and $\sum C_n(1-C_n) = \infty$. If $x_1 \varepsilon K$ and $x_{n+1} = c_n T x_n + (1-C_n)x_n$ for $n \geq 1$, then $\{x_n\}$ converges weakly to a fixed point of T. We have also obtained a similar result under a weaker assumption that space is uniformly convex Banach space with Frechet differentiable norm. thus we have generalised the theorem of Reich [4]. Ilic and Rakocavic [1] have discussed about common fixed f maps on cone metric space. Also we have discussed an application of the iteration scheme to obtain an approximate solution of the system of equation. We have also investigated mappings which are not necessarily contraction but give fixed point. We have also given an example of continuous function f which satisfy the condition

(A) $\left[d(f^{n(x)}(Y)f^{n(x)}(X)\right] \leq kd(y,x)$

where (X, d) is a complete metric space, x ε X, y ε Y but not a contraction.

2. Basic definition and theorems

Definition 1:

Let (X, d) be a metric space and f: x→X a mapping. If f satisfies the condition

$$d[f(x), f(y)] \leq K\, d(x,y)$$

For all x, y ε X and for some K < 1 then f is called a contraction.

Definition 2:

Let (X, d) be a complete metric space. T a self map of X. Let $x_0 \varepsilon X,\ X_{n+1} := f(T, x_n)$ denote an iteration procedure which yields a sequence of points $\{x_n\}$. Suppose that $\{x_n\}$ converges to a fixed point p of T.

Let $\{y_n\} \subset X,\ \varepsilon_n := d[y_{n+1} f(T, y_n)]$. If lim $\varepsilon_n = 0$ implies that lim $y_n = p$, then the iteration procedure is said to be T-stable.

The contractive definition we shall use is the following. Suppose there exists a constant c, 0 ≤ c <1 such that, for each x, y ε X.

$$\| T_x - T_y \| \leq c \ \max \left\{\| x-y \|, \left[\| x-T_x \| + \| y-T_y \|\right]/2 \ . \| x-T_y \|, \ \| y-T_x \|\right\}$$

Theorem 1:

Let K be a closed convex subset of a uniformly convex Banach space E with a Frechet differentiable norm $\{T_i$: i=1, 2, ……k } a family of non-expansive self-mappings of K with a non-empty set of common fixed points and $\{C_n\}$ a real sequence such that $0 \leq C_n \leq 1$ and $\sum_{n=1}^{\infty} C_n(1-C_n) = \infty$. If $x_1 \varepsilon K$ and $x_{n+1} = (1-C_n)x_n + C_n\, T_k\, U_{k-1}\, x_n$ for n ≥1, then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i$: i=1, 2, ……k }.

Proof:

U_j and $T_j U_{j-1}$, j = 1, 2, ……k are non-expansive and map k into itself and the families

$\{ U_1 , U_2 , ………., U_k \}$

and $\{ T_1 , T_2 , ………., T_k \}$

have the same set of common fixed points. Since $T_k U_{k-1}$ is a non-expansive self-mapping of k and the sequence $\{x_n\}$ defined by $x_{n+1} = (1-c_n)x_n + c_n T_k U_{k-1} x_n$ is of the same form as $x_{n+1} = (1-c_n)x_n - c_1 T_1 x_n$ where k = 1, so it converges weakly to a fixed point V of $T_k \ U_{k-1}$ by the theorem of Reich [4] we shall next show that V is a common fixed point of T_k and $U_{k-1}(k \geq 2)$.

To this end we first show that $T_{k-1} \ U_{k-2} \ V(k \geq 2)$. Suppose not, let $z = U_{k-1} \ V = (1-\alpha)v + \alpha \ T_{k-1} \ U_{k-2} \ v$. Then z ≠ v. By hypothesis, there exists a point w such that

$T_1 \ w = T_2 \ w = ... T_k \ w = w.$ Since $\{T_i\}$ and $\{U_i\}$ have the same common fixed points, it follows that $T_{k-1} \ U_{k-1} \ w = w$.

By non-expansiveness

$$\|T_{k-1} \ U_{k-2} \ v - w\| \leq \| \ v - w \| , \qquad ….. \qquad (1)$$

and $$\| \ T_k \ \ z - w \| \leq \| \ z - w \|$$

Again $T_k \ \ z = T_k \ U_{k=1} \ \ v = v$.

Since uniformly convex Banach space is strictly convex, it follows that

$$\begin{aligned} \| \ v - w \| &\leq \| \ \ z - w \| \\ &= \| \ (1-\alpha)v + \alpha \ T_{k-1} \ T_{k-2} \ v - w \| \\ &= \| \ (1-\alpha)\,(v-w) + \alpha \,(T_{k-1} U_{k-2} \ v - w \| \\ &< \ \max \ \{\| \ v - w \|, \| \ T_{k-1} \ U_{k-2} \ v - w \|\} \end{aligned}$$

which contradicts (1). $T_{k-1} U_{k-2} \ V = V.$ As $U_{k-1} = (1-\alpha)I + \alpha T_{k-1} \ U_{k-2}$, we have $U_{k-1} \ v = (1-\alpha)v + \alpha v = v$ and $v = T_k U_{k-1} \ v = T_k v$. Thus v is a common fixed point of T_k and U_{k-1} . Since $T_{k-1} U_{k-2} \ v = v,$ we repeat the above argument to show that $T_{k-2} \ U_{k-3} \ v = v$ and consequently v must be a common foxed point of T_{k-1} and U_{k-2} . Continuing in this manner, we can prove that $T_1 \ U_0 \ v = v$ and that v is a common fixed point of T_2 and U_1 . Hence v is a common fixed point of $\{T_i \ : i = 1,2,…..,k)$.

REMARK 1 : As an application of the above theorem we have the following:

Suppose we have a system of equations of the form

$$x - F_i x = f_i \ , \ i = 1,2,.......k \ , \qquad (2)$$

where F_i is a non expansive self mapping of E and each f_i is a given element of E. Consider the family mappings defined by $T_i x = f_i + F_i x, \ \ i = 1,2,.......k.$

Each T_i is a non-expansive self mapping of E. Again x is a solution of (2) if x is a common fixed point of $\{T_i\}$. Since our theorem 1 remains valid for K=E, the iteration scheme can be applied to obtain an approximate solution of the above system of equations.

THEOREM 2:

Let (X,d) be a complete metric space, and $f : \ X \to X$ a continuous mapping satisfying the condition : there exists a k < 1 such that for each $x \ \varepsilon \ X$, there is a positive integer n(x) such that for all $y \ \varepsilon \ X$

(A) $$d\left(f^{\,n(x)}(y), \ \ f^{\,n(x)}(x)\right) \leq \ kd \ \ (y, \ x)$$

Then f has a unique fixed point u and $f^{\,n} \ (x_0) \to u$ for each $x_o \ \varepsilon \ X.$

LEMMA: If $f : \ X \to X$ be any mapping satisfying the condition of the above theorem then for each $x \, \varepsilon \, X$, $r(x) = \sup_n \ d\left(f^{\,n}(x), \ x\right)$ is finite.

PROOF OF THE THEOREM :

Let $x_o \; \varepsilon \; X$ be arbitrary. Let $m_o = n(x_o)$, $x_1 = f^{m_o}(x_o)$ and inductively $m_i = n(x_i)$, $x_{i+1} = f^{m_1}(x_i)$ we show that the sequence $\{x_n\}$ is a convergent sequence. By routine calculation we have

$$d(x_{n+1}, x_n) = d\left(f^{m_{n-1}} . f^{m_n}(x_{n-1}); \quad f^{m_{n-1}}(x_{m-1})\right)$$
$$\leq kd\left(f^{m_n}(x_{n-1}), \quad x_{n-1}\right)$$
$$\leq ...... \leq k^n \, d\left(f^{m_n}(x_0), \quad x_0\right)$$

Therefore, it follows by lemma that $d(x_{n+1}, x_n) \leq k^n r(k_0).$ Thus, for m > n

$$d(x_m, x_n) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i)$$
$$\leq \frac{k^n}{1-k} r(x_o) \to 0 \quad as \quad n \to \infty.$$

The sequence $\{x_n\}$ is therefore Cauchy. Let $x_n \to u \; \varepsilon \; X$ If $f(u) \neq u$, then there exists a pair of disjoint closed neighbourhoods U and V such that $u \; \varepsilon \; U$, $f(u) \; \varepsilon \; V$ and

$$\rho = Inf \{ d(x, y) : x \, \varepsilon \, U, \; y \, \varepsilon \, V \} > 0. \qquad (3)$$

Since f is continuous, $x_n \; \varepsilon \; U$ and $f(x_n) \; \varepsilon \; V$ for all n sufficiently large. However,

$$d(f(x_n), x_n) = d\left(f^{m_{n-1}} . f(x_{n-1}), \; f^{m_{n-1}}(x_{n-1})\right)$$
$$\leq kd\left(f(x_{n-1}), \; x_{n-1}\right) \leq ....$$

$< k^n \, d\left(f(x_0), \; x_0\right) \to 0$ as $n \to \infty$, contradiction (3). Thus $f(u) = u$

The uniqueness of the fixed point follows immediately from (A).

To show that $f^n(x_0) \to u$, set

$$\rho_* = \max \left\{ d\left(f^m(x_0), u\right) : \; m = 0,1,2,......, (n(u)-1) \right\}$$

If n is a sufficiently large integer, then

$n = r . n(u) + q, 0 \leq q < n(u), \; r > 0$ and

$$d\left(f^n(x_0), u\right) = d\left(f^{r.n(u)+q} \; (x_0) f^{n(u)}(u)\right)$$
$$\leq kd\left(f^{(r-1).n(u)+q(x_o)}, \; u\right)$$
$$\leq .... < k^r \, d\left(f^q(x_0), \; u\right) \leq k^r q_*$$

Since $n \to \infty$ implies $r \to \infty$, we have

$d\left(f^n(x_0), u\right) \to 0$ as $n \to \infty$, This establishes the theorem.

Here we give an example of a continuous function f which satisfies (A) but is not a contraction.

EXAMPLE :

Let X be the closed unit interval [0, 1] with the usual metric. Write

$$X = \bigcup_{n=1}^{\infty} \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] U \{0\},$$

and let f : X → X be defined as follows.

For each n = 1,2,……, let

$$f : \left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right] \to \left[\frac{1}{2^{n+11}}, \frac{1}{2^n}\right]$$

be defined by

$$f(x) = \frac{n+2}{n+3}\left(x - \frac{1}{2^{n-1}}\right) + \frac{1}{2^n},$$

if $x\varepsilon\left[\frac{3n+5}{2^{n+1}(n+2)}, \frac{1}{2^{n-1}}\right]$

$$= \frac{1}{2^{n+1}} \quad if \quad x\varepsilon\left[\frac{1}{2^n}, \frac{3n+5}{2^{n+1}(n+2)}\right]$$

and let f(0) =0.

It is obvious that f is a non-decreasing, continuous function on (0,1) with 0 as the only fixed point, and that f is not a contraction. If $x\varepsilon\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ and $y \varepsilon X$, then by a routine examination of cases $y\varepsilon\left[\frac{1}{2^m}, \frac{1}{2^{m-1}}\right]$ for m ≥ n and m ≤ n, it is easy to verify that f satisfies

$|f(x) - f(y)| \leq \frac{n+3}{n+4}|x-y|$ for all $y \varepsilon X$. Therefore, if we choose k = ½ in (A), then for each $x\varepsilon\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$, $n(x)$ may be taken as n+3, whereas n(0) may be taken as any integer greater than one.

To show that the condition in [A] is stronger than (A), let 0 ≤ k <1, and N (a natural number) be given ; it will be shown that there exists x and y such that $\left|f^N(x) - f^N(y)\right| > k\,|x-y|$. Choose and fix $n > \left(\frac{Nk}{1-k}\right) - 2$. Since f^i is uniformly continuous on [0,1] for i= 1,2,….,N, there is some δ > 0 such that $|x-y| < \delta \Rightarrow \left|f^i(x) - f^i(y)\right| < \frac{n+N+3}{(n+N+2)2^{n+N+1}}$ for i = 1,2,……,N. Setting $x = \frac{1}{2^{n-1}}$ and y any member of $\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]$ such that 0 < | x – y | < δ, it can be shown that $f^i(x)$ and $f^i(y)$ are both members of

$$\left[\frac{3(n+i)+5}{2^{n+i+1}(n+i+2)}, \frac{1}{2^{n+i-1}}\right]$$

for i = 1,2,…..,N. Thus

$$|f(x) - f(y)| = \frac{n+2}{n+3}\,|x-y|,$$

$$\left|f^2(x) - f^2(y)\right| = \frac{n+2}{n+4}\,|x-y|, \ldots\ldots, \left|f^N(x) - f^N(y)\right|$$

$$= \frac{n+2}{n+2+N}|x-y| > k\,|x-y|.$$

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