Volume-04
Issue-05
May-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

New Approximate Fixed Point Theorems

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ARTICLE DETAILS

Article History
Published Online: 25 May 2019

Keywords
Fixed point Theorems

ABSTRACT

The purpose of this paper is to construct an iteration scheme for approximating a fixed point of non-expansive non-self maps and to prove some strong and weak convergence theorems for such maps. Our theorems improve and generalize some previous results.

1. Introduction

Let K be a non-empty Banach space E, which is also a non-expansive retract of E. Let T : K → E be a non-expansive mapping. The following iteration scheme is studied.

$$x_{n+1} = P(1-\alpha_n)x_n + \alpha_n TP[(1-\beta_n)x_n + \beta_n Tx_n] \qquad (1)$$

with $x_n \in K, n \geq 1, \{\alpha_n\}, \{\beta_n\}$ are sequences in [0, 1] and P is a non-expansive retraction of E and K.

Theorem – 1:

(Tan and Xu[1]), Suppose E be a uniformly convex Banach space which satisfies Opial's condition or has a Frechet differentiable norm and K, a non-empty closed convex bounded subset of E. Let T : K → K be a non-expansive mapping. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences in [0, 1] such that

$$\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty, \qquad \sum_{n=1}^{\infty} \beta_n(1-\alpha_n) = \infty < \infty,$$

$$\lim \quad \sup_{n\to\infty} \quad \beta_n < 1 .$$

Then the sequence $\{x_n\}$ generated from arbitrary $x_n \in K$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n \; T[(1-\beta_n)x_n + \beta_n \; Tx_n], \quad n \geq 1, \qquad (2)$$

converges weakly to some fixed point of T.

Definition - 1

Let E be a real Banach space; A subset K of E is said to be a retract of E if there exists a continuous map P:E →K such that

$$Px = x \qquad \text{for all } x \in K.$$

Definition – 2:

P : E → E is said to be retraction if
$P^2 = P$.

It follows that if a map P is a retraction then,

Py = y for all y in the range of P.

Definition – 3 : A set K is said to be optimal if each point outside K can be moved closer to all points of K.

Definition – 4 : If E is a separable, strictly convex, smooth, reflexive Banach space, and if K=E is an optimal set with interior, then K is a non-expansive retract of E.

Definition – 5 : A subset of l_p, with l < P < ∞, is a non-expansive retract iff it is optimal.

Note: Every non-expansive retract is optimal. In strictly convex Banach spaces, optimal sets are closed and convex. However, every closed convex subset of a Hilbert space is optimal and also a non-expansive retract.

Definition – 6 : A function T is said to be demiclosed at p if whenever $\{X_n\}$ is a sequence in D(T) such that $\{X_n\}$ converges weakly to x*∊ D(T) and $\{Tx_n\}$ converges strongly to p, then Tx*=p, where D(T) and R(T) is the domain and range of T.

Definition – 7 : A Banach space E is said to have the Kadec – Klee property if for every sequence $\{X_n\}$ in E, X_n → x weakly and $\| x_n \| \to \| x \|$ strongly together implying

$$\| x_n - x \| \to 0.$$

A function T : K → E be a non-expansive mapping and K be a non-empty closed convex subset of a real uniformly convex Banach space E, which is also a non-expansive retract of E.

The following iteration scheme is studied

$$x_{n+1} = P\left(\left(1-\alpha_n\right)x_n + \alpha_n\ T\ P\left[\left(1-\beta_n\right)x_n + \beta_n\ Tx_n\right]\right) \quad (3)$$

with $x_n \in K$, $n \geq 1$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] and P is a non-expansive retraction of E onto K.

Lemma – 1 :

(Xu[3]) Suppose R > 1 and p > 1 be two fixed number and E a Banach space. Then |E is uniformly convex iff ∃ a continuous, strictly increasing, and convex function g : [0, ∞) → [0, ∞) with g(0)=0 s.t.

$$\left\|\lambda x + (1-\lambda)y\right\|^P \leq \|x\|^P$$

$$+ (1-\lambda)\left\|y\right\|^P - W_P(\lambda)g(\left\|x-y\right\|) \quad \text{for all } x, y \in B_R(0)$$

$$= \left\{x \in E : \left\|x\right\| < R\right\} \quad \text{and} \quad \lambda \in [0,1]$$

where $W_P(\lambda) = \lambda(1-\lambda)P + P(1-\lambda).$

Theorem – 2 :

Suppose non-expansive mapping T : K → K with $x^* \in F(T)$ where F(T) = { x ∈ K : Tx = x}. Suppose E be a real uniformly convex Banach space and K a non-empty closed convex subset of E which is also a non-expansive retract of E. Suppose $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in [∈, 1 −∈] for some $T_n \in (0,1)$. And define the sequence $\{X_n\}$ by the iteration

$$x_{n+1} = P\left(\left(1-\alpha_n\right)x_n :\ \alpha_n\ T\ P\left[\left(1-\beta_n\right)x_n + \beta_n\ \ Tx_n\right]\right)$$

where $X_n \in K$, $n \geq 1$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] and P is a non-expansive retraction of E onto K.

Then we have to prove that

Proof: We observe that

$$\lim_{n\to\infty} \left\|x_n - x^*\right\| \ exists$$

$$\left\|x_{n-1} - x^*\right\| = \left\|P\left(\left(1-\alpha_n\right)x_n + \alpha_n TP\left[\left(1-\beta_n\right)x_n + \beta_n Tx_n\right]\right) - Px^*\right\|$$

$$\leq \left\|\left(1-\alpha_n\right)x_n + \alpha_n TP\left[\left(1-\beta_n\right)x_n + \beta_n Tx_n\right] - x^*\right\|$$

$$\leq \left(1-\alpha_n\right)\left\|x_n - x^*\right\|\ \alpha_n\ \left\|TP\left[\left(1-\beta_n\right)x_n + \beta_n\ Tx_n\right] - Tx^*\right\|$$

$$\leq \left(1-\alpha_n\right)\left\|x_n - x^*\right\| + \alpha_n\ \left\|P\left[\left(1-\beta_n\right)x_n + \beta_n\ Tx_n\right] - x^*\right\|$$

$$\leq \left(1-\alpha_n\right)\left\|x_n - x^*\right\| + \alpha_n\ \left[\left(1-\beta_n\right)\left\|x_n - x^*\right\| + \beta_n\left\|Tx_n - Tx^*\right\|\right]$$

$$\leq \left(1-\alpha_n\right)\left\|x_n - x^*\right\| + \alpha_n\ \left[\left(1-\beta_n\right)\left\|x_n - x^*\right\| + \beta_n\left\|x_n - x^*\right\|\right]$$

$$= \left\|x_n - x^*\right\|$$

and we have

$$\left|x_{n+1} - x^*\right| \leq \left\|x_1 - x^*\right\|$$

$\Rightarrow \{x_n\}$ is bounded.

Lemma – 2:

(Tan and Xu[2]): Suppose $\{\lambda_n\}$ and $\{\sigma_n\}$ be sequences of non-negative real numbers s. t.

$$\lambda_{n+1} \leq \lambda_n + \sigma_n\ .V_n \geq 1 \quad \text{and} \quad \sum_{n=1}^{\infty}\sigma_n\ <\ \infty$$

Then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence

$\{\lambda_{nj}\}$ of $\{\lambda_n\}$ such that

$\lambda_{nj} \to 0$ as $j\to\infty$, then

$\lambda_n \to 0$ as $n\to\infty$,

guarantees that

$$\lim_{n\to\infty} \left\|x_n - x^*\right\| \text{ exists.}$$

This is the required proof.

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