Volume-04
Issue-06
June-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

Ishikawa Iteration In Contractive Mapping

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ARTICLE DETAILS

Article History
Published Online: 12 June 2019

Keywords
mapping, lemma, Iteration, contractive, space.

ABSTRACT

The present paper provides some lemma and theorems related with Ishikawa Iteration in contractive mappings.

1. Introduction

A mapping T : C → C is said to be non-expansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all X, Y in C, where C be a non-empty closed bounded convex subset of a Banach space. X. Ishikawa [3] introduced a new iteration procedure for approximating fixed points of pseudo-contractive compact mappings in Hilbert spaces as

$$X_{n+1} = t_n T[s_n Tx_n + (1-s_n)x_n] + (1-t_n)x_n,\ n = 0,1,2,\ldots.. \qquad \text{.... (A)}$$

where $\{t_n\}$ and $\{s_n\}$ are sequence in [0,1] satisfying certain restrictions.

Banach space X is said to have a Frechet differentiable norm if for each x ε S(X), the unit sphere of X, the limit

$$\lim_{t \to \infty} \frac{\|x + ty\| - \|x\|}{t}$$ exists and is attained uniformly in y ε S(X). In this case we have

$$\frac{1}{2}\|x\|^2 < h, J(x) \quad \leq \frac{1}{2}\|x + h\|^2 \leq \frac{1}{2}\|x\|^2 < h, J(x) > g(\|h\|) \qquad \textbf{(1.1)}$$

For all bounded x, h ε X, where $J(x) = \partial \frac{1}{2}\|x\|^2$ is the Frechet derivative of the functional $\frac{1}{2}\|\ .\ \|^2$ at x ε X, < . , . > is the pairing between X and X* , and g(*) is a function defined on [0, ∞) such that $\lim_{t \to 0} \frac{g(t)}{t} = 0.$ Suppose T : C →C is a non expansive mapping where C be a bounded closed convex subset of a uniformly convex Banach space X. Then we write for each integer n ≥ 0,

$$T_n(x) = t_n T[s_n Tx + (1-s_n)x] + (1-t_n)x,\ x\ \varepsilon\ C, \qquad \textbf{(1.2)}$$

Then T_n : C →C is also non expansive and the Ishikawa iteration { x_n } defined by (A) can be written as

$$X_{n+1} = T_n x_n\ , \quad n = 0,1,2,\ldots.. \qquad \textbf{(1.3)}$$

We obtain $F(T_n) \supseteq F(T)$ for $n \geq 0,$ where F(T) denotes the set of fixed point of T.

2. THEOREMS AND LEMMA

Lemma 1 : Let $\{x_n\}$ and $\{y_n\}$ are two sequences of non negative members such that $x_{n+1} \leq x_n + y_n$ for all n ≥ 1. If $\Sigma_n y_n$ converges then $\lim_n$ can exists.

Lemma 2 : Let

(i) $\overline{\lim}_n\ s_n < 1;$

(ii) $\sum_{n=0}^{\infty} s_n(1-t_n) < \infty$ and

(iii) $\sum_{n=0}^{\infty} t_n(1-t_n) = \infty.$

Then prove that

$$\lim_n \|Tx_n - x_n\| = 0.$$

Lemma 3 : Let X be Banach space and in addition such that X has a Frechet differentiable norm. Then for every

$$f_1,\ f_2 \ \text{in}\ F(T)\ \text{and}\ 0 < t < 1,\ \lim_n \|tx_n + (1-f_1) - f_2\| \text{ exists.}$$

Lemma 4 :

$$\lim_n \|x_n - f\| \text{ exists for every f belong in F (T).}$$

THEOREM 1 : Let T : C → C a non-expansive mapping where C be a bounded closed convex subset of X uniformly convex Banach space X which satisfies Opial's conditions or whose norm is Frechet differentiable. Then for any initial guess x_o in C. The Ishikawa iteration process { x_n } defined by (A) with the restrictions that $\sum_{n=0}^{\infty} t_n(1-t_n)$ diverges, $\sum_{n=0}^{\infty} s_n(1-t_n)$ converges, and $\overline{\lim}_n\ s_n$ is less than one, converges weakly to a fixed point of T.

PROOF: By applying the article of F.E. Browder [2] , we obtain that if X is uniformly conved, then T has a fixed point and I – T is semiclosed at the origin that is for any sequence { y_n } in C, the conditions y_n → y weakly and y_n – Ty_n → 0 strongly imply y – Ty =0. It thus follows from Lemma 2 that $W_w(x_n) \subset F(T).$ Here $W_w(x_n)$ denotes the weak w – lim set of the sequence { x_n } that is the set { u ε X : u = weak – $\lim_{k\to\infty} x_{nk}$ } for some $n_k \uparrow \infty.$ To show that { x_n } converges weakly to a fixed point of T, it suffices to show that $W_w(x_n)$ consists of exactly one point. To this end, we first suppose that X satisfies Opial's condition and suppose p ≠ q are in $\lim_{k\to\infty} x_{nk}$. Then p = weak – $\lim_{k\to\infty} x_{nk}$ and q = weak – $\lim_{j\to\infty} x_{mj}$ for some $n_k \uparrow \infty$ and $m_j \uparrow \infty$. By Lemma 4 and Opial's condition of X, we then have

$$\begin{aligned}\lim_n \|x_n - p\| &= \lim_k \|x_{n_k} - p\| < \lim_k \|x_{n_k} - q\| \\ &= \lim_j \|x_{m_j} - q\| < \lim_j \|x_{m_j} - p\| \\ &= \lim_n \|x_n - p\|,\end{aligned}$$

arriving at a contradiction. This proves the theorem in the case in which X satisfies Opial's condition. We now assume that X has a Frechet differentiable norm. Substituting $f_1 - f_2$ and $t(x_n - f_1)$ for x and h, respectively, in (1), where f_1, f_2 ε F (T) and 0 < t < 1, we have

$$\begin{aligned}&\frac{1}{2}\|f_1 - f_2\|^2 + t < x_n - f_1,\ J(f_1 - f_2) > \\ &\le \frac{1}{2}\|tx_n + (1-t)f_1 - f_2\|^2\end{aligned}$$

$$\leq \frac{1}{2}\|f_1 - f_2\|^2 + t < x_n - f_1,\ J(f_1 - f_2) > + g(t\,\|x_n - f_1\|)$$

Applying lemma 2, we have

$$\frac{1}{2}\|f_1 - f_2\|^2 + t\,.\overline{\lim_n} < x_n - f_1,\ J(f_1 - f_2) >$$

$$\leq \lim_n \frac{1}{2}\|tx_n + (1-t)f_1 - f_2\|^2$$

$$\leq \frac{1}{2}\|f_1 - f_2\|^2 + t\,.\lim_n < x_n - f_1,\ J(f_1 , f_2) > +0(t).$$

Therefore

$$\overline{\lim_n} < x_n - f_1 , J(f_1 - f_2) > \leq \overline{\lim_n} < x_n - f_1 , J(f_1 - f_2) > +0(t)/t.$$

Let $t \to 0^+$, we note that

$$\lim_n < x_n - f_1, J(f_1 - f_2) >$$ exists.

This implies that

$$< p - q, J(f_1 - f_2) > = 0, \qquad (1.1)$$

for all p, q in $W_w(x_n) < p - q,\ in\ f_1 , f_2$ in F(T). Since $W_w(x_n) \subset F(T)$ for any p, q in $W_w(x_n)$, by replacing f_1 , f_2 in (1.1) by p, q, respectively, we have

$$\|p - q\|^2 = < p - q,\ J(p - q) > = 0$$

This means $W_w(x_n)$ must be singleton.

Again we define a mapping T : E → E such that

$$\|Tx - Ty\| \leq \max\Big\{ \|x - y\| ,\ \frac{1}{2}(\|x - Tx\| + \|y - Ty\|),\ \frac{1}{2}(\|x - Ty\| + \|y - Tx\|) \Big\} \qquad \text{...(A1)}$$

∀ X, y belong in E, where E be a non-empty bounded closed convex subset of a Banach space X. The Ishikawa type iterative process is defined by

$$X_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n Tx_n] \qquad \text{...}(I_1)$$

and x_o ε E, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of non-negagive numbers such that

$$0 \leq \beta_n \leq 1,\quad 0 < \alpha_n \leq \alpha_n < 1, \qquad \text{...}(I_1.i)$$

$$\lim_{n\to\infty} \sup\ \beta_n < 1 \qquad \text{...}(I_1.ii)$$

Rhoades [4] and (A_1) but he considered the Menn iterative process, which is defined by

$$X_{n+1} = (1-\alpha_n)x_n + \alpha_n\ Tx_n$$

x_o ε E, …(M_1)

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfies; (i) $\alpha_o = 1,$ (ii) $0 \leq \alpha_n \leq 1$ and (iii) $\Sigma\, \alpha_n = \infty.$ Bandyopadhyay and Ganguly [1] used the following contractive definition:

$$\|Tx - Ty\| \leq q_1\|x - y\| + q_2\|x - Tx\| + q_3\|y - Ty\|$$

$$+ q_4 \|x - TY\| + q_5 \|y - Tx\|$$

$$\forall\, x, y \text{ in } E, q_i \geq 0, \sum_{i=1}^{5} q_i \leq 1, \qquad \ldots(B_1)$$

THEOREM 2 : Suppose T : E → E satisfying (A_1). And Let $\{x_n\}$ be a sequence in E defined by (I_1). If $\{x_n\}$ converges then it converges to a fixed point of T, where E be a non-empty bounded closed convex subset of a Banach space X.

PROOF: Let lim x_n = p. Now

$$X_{n+1} = (1-\alpha_n)x_n + \alpha_n \; T[(1-\beta_n)x_n + \beta_n \; Tx_n]$$

that is $X_{n+1} - x_n = \alpha_n \; \{ T[(1-\beta_n)x_n + \beta_n \; Tx_n] - x_n \}$

since $X_n \to p$ and therefore

$$\lim_{n\to\infty} \{T\,[(1-\beta_n)x_n + \beta_n \; Tx_n] - x_n\} = 0$$

and

$$\{\|T\,[(1-\beta_n)x_n + \beta_n \; Tx_n] - Tx_n\|\} = 0$$

$$\leq \max \{\|\,(1-\beta_n)x_n + \beta_n \; Tx_n - x_n\|\},$$

$$\frac{1}{2}[\|\,(1-\beta_n)x_n + \beta_n \; Tx_n - T\{(1-\beta_n)x_n + \beta_n Tx_n\}\| +$$

$$\|\,x_n + Tx_n\|]$$

$$\frac{1}{2}(\|(1-\beta_n)x_n + \beta_n \; Tx_n - Tx_n\| + \|x_n - T[(1-\beta_n)x_n + \beta_n \; Tx_n]\|\,)$$

$$\leq \max \{\beta_n \|x_n - Tx_n\|, \frac{1}{2}\left(\,\beta_n \|x_n - Tx_n\| + \right.$$

$$\|\,x_n - T\,[(1-\beta_n)x_n + \beta_n \; Tx_n]\|$$

$$+ \; \|x_n - Tx_n\|\,), \frac{1}{2}\left(\,\beta_n \|x_n - Tx_n\| + \|x - Tx_n\|\right.$$

$$+ \|\,x_n - T\,[(1-\beta_n)x_n + \beta_n \; Tx_n]\|\}.$$

Now,

$$\|\,x_n - Tx_n\| \leq \|x_n - T[(1-\beta_n)x_n + \beta_n \; Tx_n]\|$$

$$+ \|T\,[\,(1-\beta_n)x_n + \beta_n \; Tx_n] - Tx_n\,\|$$

$$\leq \|\,x_n - T\,[(1-\beta_n)x_n + \beta_n \; Tx_n]\| + \max \; \{\beta_n \|x_n - Tx_n\|$$

$$\frac{1}{2}(1-\beta_n)\|\,x_n - Tx_n\,\| + \|x_n - T\,[(1-\beta_n)x_n + \beta_n \; Tx_n\,]\|$$

which yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| \; \leq 0$$

and so $\|x_n - Tx_n\| \to 0 \quad \text{as} \quad n \to \infty.$

Now,

$$\|Tx_n - T_p\| \leq \max \; \{\|x_n - p\|, \frac{1}{2}(\|x_n - Tx_n\| + \|p - T_p\|\,),$$

$$\frac{1}{2}\left(\|x_n - T_p\| + \|p - Tx_n\|\right)$$

But

$$\| p - Tp\| \leq \| p - x_n\| + \| x_n - Tx_n\| + \|Tx_n - p\|$$

$$\| x_n - Tp\| \leq \| x_n - Tx_n\| + \| Tx_n - Tp\|$$

and

$$\| p - Tx_n\| \leq \| p - x_n\| + \| x_n - Tx_n\|.$$

Hence,

$$\| Tx_n - Tp\| \leq \max \{\| x_n - p \|, 2\| x_n - Tx_n\| + \| x_n - Tp\|, 2 \| x_n - Tx_n\| + \| x_n - p\| \},$$

and

$$\lim_{n \to \infty} \| Tx_n - Tp\| = 0$$

Also

$$\| p - Tp\| \leq \| p - x_n\| + \| x_n - Tx_n\| + \|Tx_n - Tp\|.$$

Therefore

$$\| p - Tp\| \to 0 \text{ as } n \to \infty, \quad \text{then } Tp = p.$$

References

1. Bandhyopadhay and Ganguly (1991) : Soochow J. Math. 17, 269-285.
2. Browder, F.F. (1976) : Non-linear operators and non-linear equations of evolution in Banach space. Proc. Symp. Pure Math. Vol.18, Neer Math. Soc. Providence, RI.
3. Ishikawa, S. (1974) : Fixed points by a new iterative method. Proc. Neer. Math. Soc. 149, 147-150.
4. Rhoades, B.E. (1993) : Souchor, J. Math. 19, 377-80.