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Strong Convergence Theorem for A Finite Family of Asymptotically Hemicontractive Hmappings

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ABSTRACT
Here in this paper, we extend the result of theorems Qihou [1] to the iteration scheme. Further, using the result, we prove a strong convergence theorem for a finite family of asymptotically hemi contractive mappings satisfying Frum – Ketkov condition.

1. Introduction

A mapping $T : C \to C$ is said to be asymptotically pseudocontractive if there exists a sequence { s_n } with $\lim_{n\to\infty} s_n = 1$ such that

$$\|T^n x - T^n y\|^2 \le s_n\|x-y\|^2 + \|(x-T^n x)-(y-T^n y)\|^2 \quad \textbf{(1.1)}$$

for all X, Y ε C and n ε N, where C is a non-empty convex subset of a Hillbart space H.

Mapping T is called asymptotically hemicontractive if $F(T) = \{X \,\varepsilon\, C : Tx = x\} \neq \phi$ F and there exists a sequence { s_n } with $\lim_{n\to\infty} s_n = 1$ such that

$$\|T^n x - p\|^2 \le s_n\|x-p\|^2 + \|x - T^n x\|^2 \quad \textbf{(1.2)}$$

for all $X \,\varepsilon\, C, \quad p \,\varepsilon\, F(T) \quad n \,\varepsilon\, N.$

And a mapping T is called uniformly L-Lipschitzian if there exists a constant L > 0 such that

$$\|T^n x - T^n y\| \le L\,\|x-y\| \quad \textbf{(1.3)}$$

for all $X, \;\; Y \,\varepsilon\, C \quad \text{and} \quad n \,\varepsilon\, N.$

In 1991, Sahu [2] introduced the modified Ischilemma iteration method as

$$X_{n+1} = \alpha_n T^n\left[\beta_n T^n x_n + (1-\beta_n)x_n\right](1-\alpha_n)x_n,\; n = 1,2,\ldots.. \quad \textbf{(1.4)}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequence in [0,1] and the modified Mana Iteration Method as

$$X_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n,\; n = 1,2,\ldots.. \quad \textbf{(1.5)}$$

where $\{\alpha_n\}$ is suitable sequence in [0,1]

2. SOME THEOREMS AND LEMMA:

Theorem : Let C be a non-empty closed bounded and convex subset of a Hillbert space H and T : C → C completely continuous and uniformly L-Lips-chitzian and asymptotically hemicontractive with sequence

$$\{s_n\}, s_n \;\; \varepsilon\,(1, \; +\;\infty) \;\; \forall \;\; n \;\; \varepsilon \;\; N,$$

$$\sum_{n=1}^{\infty}(s_n - 1) \; < +\infty;$$

$$\{\alpha_n\}, \; \{\beta_n\} \;\; \varepsilon \;\; [0,1]; \;\; \varepsilon \; \le \; \alpha_n \le \; \beta_n \; \le \; b \quad \forall \;\; n \;\; \varepsilon \;\; N,$$

some ε > 0 and some $b \,\varepsilon\, [0, \;\; \{(1+L^2)^{1/2} - 1\}];$

$x_1 \, \varepsilon \, C \;\; \forall \;\; n \;\; \varepsilon \;\; N$ define

$$X_{n+1} = \alpha_n T^n \left[\beta_n T^n x_n + (1-\beta_n) x_n\right] (1-\alpha_n) x_n$$

Then $\{X_n\}_{n=1}^{\infty}$ converges strongly to some fixed point of T.

Suppose C be a non-empty subset of a Hillbert space H.

A mapping $T : C \to C$ is said to be hemicontractive if

$$\|Tx - p\|^2 \leq \|x - p\|^2 + \|x - Tx\|^2$$

$\forall \; x \, \varepsilon \, C$ and all $p \;\; \varepsilon \;\; F(T)$.

Das and Dabata studied the following iteration scheme for approximation a common fixed point of a finite family of hemicontractive mappings

$$\{T_j \;,\;\; j = 1,2,.....k\}$$

$$X_{n+1} = (1-\alpha_n) x_n + \alpha_n \;\; T_k \;\; Y_{k-1}(n)$$

$$y_0(n) = x_n,\; y; \;\; (n) = (1-\beta_n) x_n + \beta_n T_j y_{j-1}(n)$$

$$x_1 \;\; \varepsilon \;\; C, \qquad \textbf{(2.1)}$$

for each j = 1,2,…..(k-1) and all n ε N where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequence in [1,0].

How we introduce a modification of the above iteration scheme (2.1) for a finite family of asymptotically hemi-constructive mappings

$\{T_j \;,\;\; j = 1,2,.....,k\}$ as follows:

$$X_{n+1} = a_n x_n + b_n \;\; T_k^n \;\; y_{k-1}(n) + c_n \;\;\; u_n \;, \qquad \textbf{(2.2)}$$

$$y_0(n) = x_n,\; y; \;\; (n) = a'_n \, x_n + b'_n \, T_j^n \;\; y_{j-1}(n) + c'_n \, v_n$$

for each j = 1,2,…..,(k-1) and n ε N where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$, are suitable sequence in [0,1] s.t.

$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ and $\{u_n\}$, $\{v_n\}$ is bounded sequence in C.

This scheme contains the modified Mann and Ishikawa Iteration method with errors in the sense of Xu [3] : for k = 2, $T_1 = T_2$, our scheme reduces to the modified Ishikawa iteration method with errors as

$$X_{n+1} = a_n x_n + b_n \;\; T^n \;\; y_n + c_n \;\;\; u_n$$

$$y_n = a'_n \, x_n + b'_n \;\; T^n \;\; x_n + c'_n \;\;\; v_n \qquad \textbf{(2.3)}$$

$x_1 \, \varepsilon \, C$ for all n ε N, then (2.3) reduces to the modified Mann iteration method with error as

$$X_{n+1} = a_n x_n + b_n \;\; T^n \;\; x_n + c_n \;\;\; u_n \;\;\;, \qquad \textbf{(2.4)}$$

for all n ε N. Moreover, if $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in [0,1] and we get $b_n = \alpha_n$,

$a_n = (1-\alpha_n),\, b'_n = \beta_n,\, a'_n = (1-\beta_n) c_n = c'_n = 0$ in the above iteration methods (2.3) and (2.4), we obtain the modified Ishikawa and Mann iteration methods as (2.3) an (2.4) respectively.

Lemma – 1 : If $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\sum_{n=1}^{\infty} \phi_n < \infty$, then

$\lim_{n \to \infty} \phi_n = 0,$ where $\{\phi_n\}$ and $\{\phi_n\}$ be a sequence of non negative numbers such that for some real numbers

$$N_0 \;\; \geq 1,\;\; \phi_{n+1} \;\; \leq (1-\delta_n)\, \phi_n + \phi_n \;\; \forall \;\; n \geq N_0 \quad \text{and} \quad \delta_n \;\; \varepsilon\, [0,1].$$

Lemma – 2 : Let the family of maps $\{T_j$, j=1,2,….,k$\}$, k $\geq$ 2 satisfy

$\left\|T_i^n x - T_j^n y\right\| \leq L \|x - y\|$ for all x, y ε C, n ε N, and all pairs (i, j), L being a positive constant. Let $(X, \;\|\;\; . \;\;\|)$ be a normed linear space and $\phi \neq C \subset C$, C is convex. Also suppose the sequence $\{x_n\}_{n=1}^{\infty}$ in C be defined by

$$X_{n+1} = a_n x_n + b_n T_k^n \;\; y_{k-1}(n) + c_n \; u_n$$

$$x_1 \ \varepsilon \ C;$$

$$y_n(n) == x_n; \ y_j(n) = a_n x_n + b_n \ T_j^n \ y_{j-1}(n) + c_n \ v_n,$$

j= 1,2,….(k-1), n ε N where $\{u_n\}$, $\{v_n\}$, are arbitrary sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are real sequence in [0,1] satisfy $a_n = b_n + c_n = a'_n + b'_n + c'_n = 1$ for all n ε N. Set

$$c_n(i) = \left\| T_i^n y_{i-1}(n) - x_n \right\| \quad \text{for all n, i ε N. Then}$$

$$\| x_n - T_1 x_n \| \leq (1+2L^2) \ Ne_{n(1)} + L(1+4L)Ne_{n-1(1)}$$
$$+ (N + 2L^2 + 2L^2 N) \ Dc_n + L(N + 4LN + 2L)Dc_{n-1}$$
$$+ L(1+2L) \ Dc_{n-1}, \quad \text{for all } n \geq 2.$$

where $N = L^{k-1} + L^{k-1} + \ldots\ldots + L + 1 < \infty$ and D : = diam (C).

Lemma – 3 : If $\sum_{n=1}^{\infty} \phi_n$ is convergent and $\{\phi_n\}_{n=1}^{\infty}$ has a subsequence $\{\phi_{n_i}\}_{i=1}^{\infty}$ converging to zero. Then $\lim_{n \to \infty} \phi_n = 0$, where $\{\phi_n\}$ and $\{\phi_n\}$ be a sequence of non negative real numbers satisfy that

$$\phi_{n+1} \leq \phi_n + \phi_n, \quad \forall \ n \ \varepsilon \ N.$$

Theorem -1 : Let $\{T_j$, j=1,2,….,k$\}$, k ≥ 2 a family of asymptotically hemiconstractive selfmap of C with sequence $\{s_{n(j)}\}_{n=1}^{\infty}$ satisfy that for each j = 1,2,……,k $s_{n(j)} \geq 1 \ \forall \ n \ \varepsilon \ N$, and $\sum_{n=1}^{\infty} (w_n - 1) < \infty$ where $w_n = \max_{1 \leq j \leq k} \{s_{n(j)}\} \geq 1 \ \forall \ n \ \varepsilon N$ and C be a non-empty closed bounded and convex subset of a Hillbert space. H. Suppose that $\{T_j\}$ have at least one common fixed point in C.

Let the family of maps $\{T_j\}$ satisfy

$\left\| T_i^n x - T_j^n y \right\| \leq \ L \left\| x - y \right\|$ for all x, y ε C, n ε N, and all pairs (i, j), L being a positive constant. Let the sequence $\{x_n\}_{n=1}^{\infty}$ in C be defined by

$$X_{n+1} = a_n x_n + b_n T_k^n \ y_{k-1}(n) + c_n \ u_n \ ;$$

$$y_n(n) = x_n; \ y_j(n) = a'_n \ x_n + b'_n \ T_j^n \ y_{j-1}(n) + c'_n \ v_n,$$

j= 1,2,….(k-1), n ε N

x_1 ε C m, where $\{u_n\}$, $\{v_n\}$, are arbitrary sequences in C and $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a'_n\}$, $\{b'_n\}$ and $\{c'_n\}$ are real sequence in [0,1] satisfying the following conditions,

$$a_n + b_n + c_n = a'_n + b'_n + c'_n = 1, \ \forall \ n \geq 1,$$

$$\varepsilon \leq \alpha_n \leq \beta_n < b \quad \text{for all } n \geq 1, \text{ some } \varepsilon > 0 \text{ and some } b > 0 \text{ satisfy}$$

$$1 - 2b - L^2 \ b^2[1 + (L^2 b) + \ldots\ldots + (L^2 b)^{k-2}] > 0$$

where $\alpha_n = b_n + c_n$ and $\beta_n = b'_n + c'_n$;

$$\sum_{n=1}^{\infty} c_n < \infty \ ; \ \sum_{n=1}^{\infty} c'_n < \infty \ ,$$

Then (i) $\lim_{n \to \infty} \|x_n - T_1 x_n\| = 0$

(ii) If T_1 is completely continuous, it follows that $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_k$.

Using lemmas and definitions, the theorem can be proved easily.

REMARKS: Note that $\lim_{n \to \infty} w_n = 1$, we may assumed that $w_n \leq M_o$ for all $n \geq 1$ and some constant $M_o > 0$.

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