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Some New Results for Generalised Contraction Involving Fixed Point Properties

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ABSTRACT

The present research paper provides some new results on fixed points by Cebysev – Centre method applied to bounded sequence in normed linear space.

1. Introduction

In this paper we have some new results on fixed points by Cebysev – Centre method applied to bounded sequence in normed linear space. We have got fixed point theorems under boundary conditions of inwardness type for non-expansive mappings and generalized contractions Kirk [2,3,4] having domains which are not necessarily convex. We have also done a fixed point theorem for a generalized contraction map developed by Altman [1]. This theorem improves a result due to Horris and Sehgal [5].

2. Definitions And Results

Definition 1:

Let $(E,\|\quad\|)$ be a n.l.s. for bounded sequence $(u_n)_{n\,\varepsilon\,\mathrm{N}}\ \varepsilon\ E^{\mathrm{N}}$. We define the map $R((u_n)_{n\,\varepsilon\,\mathrm{N}}):\ E\to R^+$ by

$$R((u_n)_{n\,\varepsilon\,\mathrm{N}})(x)\ :=\overline{\lim}\,(\|u_n-x\|)_{n\,\varepsilon\,\mathrm{N}}$$

If x is a nonvoid subset of E we call

$$C(x,(u_n)_{n\,\varepsilon\,\mathrm{N}})\ :=\Big\{x\,|\,x\ \varepsilon\ X\ R((u_n)_{n\,\varepsilon\,\mathrm{N}})(x) = \inf_{y\varepsilon X} R((u_n)_{n\,\varepsilon\,\mathrm{N}})(Y)\Big\}$$

The Cebysev-center of $(u_n)_{n\,\varepsilon\,\mathrm{N}}$ with respect to X.

Definition 2 :

Let $(E,\|\quad\|)$ be a n.l.s., $\phi \neq X \subset E$ and $f : E \to E$.

(i) f is said to be nonexpansive : $\Leftrightarrow \underset{x,y\ \varepsilon\ X}{\forall} \|f(x)-f(y)\|$

$$\leq \|x-y\|.$$

(ii) f is said to be a generalized contraction : $\Leftrightarrow$

$$\underset{\alpha : x\to[0,1)}{\exists}\ \underset{x,y\ \varepsilon\ X}{\forall} \|f(x)-f(y)\| \leq \alpha(x)\ \|x-y\|$$

Definition 3 :

Let $(E,\|\quad\|)$ be a n.l.s., and $X \subset E$. X is said to be shrinkable : $\Leftrightarrow$ [0,1) $\overline{X} \subset \operatorname{int}(X).$ A convex neighborhood of the origin is a shrinkable set (but converse is not true). A non-void shrinkable set is a star shaped neighborhood of the origin (but the converse is not true).

Definition 4:

Let X be a complete metric space and $f : x \to x$ a generalized contraction, i.e.,

$$\mathrm{d}(\mathrm{fx},\ \mathrm{fy}) \leq \mathrm{Q}(\mathrm{d}(\mathrm{x},\mathrm{y})) \quad \text{for all } x, y\ \varepsilon\ X,$$

where Q satisfies the following :

(a) $0 < Q(t) < t,$ for all t ε (0, t_1],

(b) $g(t) = \frac{t}{(t - Q(t))}$ is non-increasing,

(c) $\int_0^{t_1} g(t)\ dt < \infty$

and

(d) Q is non-decreasing.

Then f has unique fixed point.

LEMMA 1 :

Let $(E, \|\quad\|)$ be a n.l.s., ϕ ≠ X ⊂ E, f : x →E and $(u_n)_{n\,\varepsilon\,N}\, \varepsilon\, E^N$ be a bounded sequence.

Then

(i) Either $C(x, (u_n)_{n\,\varepsilon\,N}) \subset C(E, (u_n)_{n\,\varepsilon\,N})$ or

$$C(x, (u_n)_{n\,\varepsilon\,N}) \subset \partial X.$$

(ii) X being boundedly weakly compact (i.e., the intersection of X with every closed ball about the origin is weakly compact) implies $C(x, (u_n)_{n\,\varepsilon\,N}) \neq \phi.$

(iii) If u_n ε X for n ε N and there is k ε Z^+

Such that

$$\lim\ (d(u_{n+k}, \infty(\{f(u_v) \mid v\,\varepsilon\,N \ \vee\ \geq n\})))_{n\,\varepsilon\,N} = 0$$

Then for every z ε X and λ ε [0,1] such that

$$\underset{y \varepsilon X}{\forall} \|f(y) - f(z)\| \ \leq\ \lambda \|y - z\|$$

We have

$$R((u_n)_{n\,\varepsilon\,N})\ (f(z))\ \leq\ R((u_n)_{n\,\varepsilon\,N})(z).$$

(iv) $(E, \|\quad\|)$ being uniformly convex in every direction (u.c.e.d)

i.e. $$\underset{z\,\varepsilon\,E}{\forall}\quad \underset{M, \Sigma > 0}{\forall}\quad \underset{\delta\,\varepsilon\,(0,1)}{\exists}\quad \underset{x, y\,\varepsilon\,E}{\forall}\quad \underset{t\,\varepsilon\,R}{\forall} \|x\|,\ \|y\| \leq M \wedge \|z\| = 1$$

$$\wedge\ x - y = tz \wedge \|t\| \geq \Sigma \Rightarrow \frac{1}{2}\|x + y\| \leq (1 - \delta)\max\{\|x\|, \|y\|\},$$

and X being convex implies card $C(x, (u_n)_{n\,\varepsilon\,N}) \leq 1.$

LEMMA 2 :

Let $(E, \|\quad\|)$ be a n.l.s., X ⊂ E be weakly compact u ε X, $(u_n)_{n\,\varepsilon\,N}\, \varepsilon\, X^N$ and f : X → E be a generalized contraction such that

$$\lim\ (u_n)_{n\,\varepsilon\,N} = u \quad \text{(weakly)} \qquad (*)$$

$$\lim\ (u_n - f(u_n))_{n\,\varepsilon\,N} = 0 \quad \text{(strongly)} \qquad (**)$$

Let furthermore one of the following conditions be satisfied:

$$f\,[\,\partial X\,] \subset X, \qquad (1)$$

f is weakly sequentially continuous, (2)

$(E, \|\ \ \|)$ is uniformly convex and

X is convex, (3)

$$\underset{(x_n)_{n\,\varepsilon\,N}\,\varepsilon\,E^N}{\forall}\ \underset{\varepsilon\,x,\,y\,\varepsilon\,E}{\forall}\ \Big[\lim(x_n)_{n\,\varepsilon\,N} = x(\text{weakly})$$

$$\Rightarrow \lim\left(\|x_n - x\|\right)_{n\,\varepsilon\,N} \le \lim\left(\|x_n - y\|\right)_{n\,\varepsilon\,N}\Big] \qquad (4)$$

(If (4) holds, $(E, \|\ \ \|)$ is said to be satisfy a weak OPIAL – condition). Then $f\,(u_n)_{n\,\varepsilon\,N} = u$ strongly.

Theorem 1 :

Let $(E, \|\ \ \|)$ be a n.l.s. which is u.c.e.d. Let X $\subset$ E be weakly compact and f : X → E be non-expansive such that

$$\underset{x\,\varepsilon\,\partial X}{\forall}\ \underset{\lambda\ \varepsilon\,(0,1)}{\exists}\ \lambda x + (1-\lambda)\,f(x)\ \varepsilon\,X, \qquad (1.1)$$

$$\bigcap_{n\,\varepsilon\,N} \operatorname{dom}(f^{\,n}) \ne \phi\ , \qquad (1.2)$$

Then Fix (f) ≠ ϕ.

PROOF: We remark first that $(E, \|\ \ \|)$ being u.c.e.d. implies.

$$\underset{z\,\varepsilon\,E}{\forall}\ \underset{\lambda\,\varepsilon\,(0,)}{\forall}\ \underset{M,\varepsilon>0}{\forall}\ \underset{\delta\,\varepsilon\,(0,1)}{\exists}\ \underset{x,y\,\varepsilon\,E}{\forall}\ \underset{t\,\varepsilon\,R}{\forall}\Big[\|x\|,\ \|y\| \le \|z\| = 1$$

$$\le M \wedge X - y = tz\ |t| \ge \varepsilon \Rightarrow$$

$$\Rightarrow \|\lambda x + (1-\lambda)y\| \le (1-\delta)\max\{\|x\|,\ \|y\|\}\Big]$$

as is easily seen by the inequality

$$\|\lambda x + (1-\lambda)y\| \le 2\lambda\frac{1}{2}\|x+y\| + (1-2\lambda)\max\{\|x\|,\ \|y\|\}$$

where x, y ε E and $\lambda\ \varepsilon\,(0,\ \frac{1}{2}]$ (without loss of generality). Choose $x\ \varepsilon \bigcap_{n\,\varepsilon\,N}\operatorname{dom}\left(f^{\,n}\right)$ and then fix y ε C (x, f^n $(x)_{n\,\varepsilon\,N}$). Assume now f(y) ≠ y.

By (1.1) there is λ ε (0,1) such that

v : = λy + (1–λ) f(y) ε X. If we set $\varepsilon := \|x - f(y)\|$,

z : = ε^{-1} (y – f(y)) and M : = diam (f [X] U X), there is by the remark above a δ ε (0,1) such that for n ε N

$$\|v - f^{\,n+1}(x)\| = \|\lambda(y - f^{\,n+1}(x) + (1-\lambda)(f(y) - f^{\,n+1}(x))\|$$

$$\le (1-\delta)\max\left\{\|y - f^{\,n+1}(x)\|\ ,\ \|y - f^{\,n}(x)\|\right\}$$

This implies

$$R\left((f^{\,n}(x))_{n\,\varepsilon\,N}\right)(v) \le (1-\delta)\ \inf_{w\,\varepsilon\,X} R\left((f^{\,n}(x))_{n\,\varepsilon\,N}\right)(w)$$

a contradiction to f(y) ≠ y.

Theorem 2 :

Let $(E, \|\ \ \|)$ be a n.l.s. which is u.c.e.d. Let X be a non-void weakly compact subset of E and f : X → E be non-expansive such that f [∂X] ⊂ int (x). Then Fix (f) ≠ ϕ.

PROOF: Let u_0 ε X. A simple observation yields the existence of a sequence

$(u_n)_{n\,\varepsilon\,N}\,\varepsilon\,X^N$ such that

$$\underset{n\,\varepsilon\,N}{V}\left[(f(u_{n-1})\ \varepsilon\, X \Rightarrow u_n = f(u_{n-1})) \wedge (f(u_{n-1}) \notin X \Rightarrow u_n\ \varepsilon\ \partial X \cap co(\{u_{n-1}, f(u_{n-1})\}))\right].$$

It is easy to see that we have for

$$n\ \varepsilon\, Z^+ : u_{n+2}\ \ \varepsilon\ \ co\ \ (\{f(u_n\ , f(u_{n+1})\})$$

which yields – by lemma 1 (iii) – for every

$z\ \varepsilon X : R((u_n)_{n\,\varepsilon\,N})\ f(z) \le R((u_n)_{n\,\varepsilon\,N})\ (z)$. This implies – observing f [∂ X] ⊂ int (X) – That (X, C $(u_n)_{n\,\varepsilon\,N}$) (z). This implies – observing

f [∂ X] ⊂ int (X) – that C (X, C $(u_n)_{n\,\varepsilon\,N}$) is not a subset of ∂X. Therefore by lemma 1 (i)

$C\ (X,\ (u_n)_{n\,\varepsilon\,N}) \subset C\ (E,\ (u_n)_{n\,\varepsilon\,N})$. Choose now

$x\ \varepsilon\ C\ (X,\ (u_n)_{n\,\varepsilon\,N})$. We have shown that

$\{x, f(x)\} \subset C\ (E,\ (u_n)_{n\,\varepsilon\,N})$ and hence f(x) = x

(by lemma 1 (iv)).

Theorem 3 :

Let $(E, \|\ \ \|)$ be a n.l.s., X ⊂ E be boundedly weakly compact and f : X → E be a generalized contraction such that

$$\underset{x\,\varepsilon\,\partial X}{\forall}\ \ \underset{\lambda\ \varepsilon[0,1]}{\exists}\ \ \lambda x + (1-\lambda) f(x)\ \varepsilon X, \qquad (3.1)$$

$$\bigcap_{n\,\varepsilon\,N} dom\,(f^n) \neq \phi\ , \qquad (3.2)$$

Then there is a unique x ε X with f(x) = x and for $y\,\varepsilon \bigcap_{n\,\varepsilon\,N} dom\,(f^n)$ we have

$$\lim (f^n(y))_{n\,\varepsilon\,N} = x\ \text{(strongly)}.$$

PROOF: Let $y\,\varepsilon \bigcap_{n\,\varepsilon\,N} dom\,(f^n)$. Because for n ε N the inequality $\|f^n(y) - y\| \le (1-\alpha(y))^{-1}\,\|f(y)-y\|$ is satisfied, the sequence $(f^n\ (y)_{n\,\varepsilon\,N})$ is bounded and therefore – by lemma 1(ii) - $C\ (X, f^n\ (y)_{n\,\varepsilon\,N}) \neq \phi.$

Choose x ε $C\ (X, f^n\ (y)_{n\,\varepsilon\,N})$ and then ε [0,1) such that $\lambda x + (1-\lambda)\ f(x)\ \varepsilon\ X.$ We immediately find – using lemma 1 (iii)

$$\begin{aligned} R((f^n(x))_{n\,\varepsilon\,N})\ (x) &\le R((f^n(x))_{n\,\varepsilon\,N})\ (\lambda x + (1-\lambda) f(x)) \\ &\le \lambda R((f^n(x))_{n\,\varepsilon\,N})\ (x) \\ &\quad + (1-\lambda)\, R((f^n(y))_{n\,\varepsilon\,N})\ (f(x)) \end{aligned}$$

$$\leq [\lambda + (1-\lambda)\,\alpha(X)]\ R\big((f^{n}(y))_{n\,\varepsilon\,N}\big)\ (x)$$

Hence $R\big((f^{n}(y))_{n\,\varepsilon\,N}\big)\ (x) = 0$ i.e.,

$\lim\ \big((f^{n}(y))_{n\,\varepsilon\,N}\big) = x$ (strongly) and f(x) = x. The uniqueness of the fixed point is trivial.

Theorem 4 :

Let $(E, \|\quad\|)$ be a n.l.s. which is u.c.e.d., $\phi \neq X \subset E$ be boundedly weakly compact and f : X → E be a generalizer contraction such that f[∂X] ⊂ X. Then there is exactly one fixed point of f and for $y\ \varepsilon \bigcap_{n\,\varepsilon\,N} \text{dom}\,(f^{n})$ the successive approximants $\big((f^{n}(y))_{n\,\varepsilon\,N}\big)$ converge strongly to this fixed point.

PROOF: Let u_o ε X and choose a sequence $(u_n)_{n\,\varepsilon\,N}\,\varepsilon\, X^{N}$ as in the proof of theorem 2. Using $u_{n+2}\ \varepsilon\, co(\{f(u_n)),\ f(u_{n+1})\})$, we immediately find the estimation

$\|u_n - u_o\| \leq (1-\alpha(u_o))^{-1}\,\|\,f(u_o) - u_o\|$. Hence $(u_n)_{n\,\varepsilon\,N}$ is bounded, which yields $C\,(X, (u_n)_{n\,\varepsilon\,N}) \neq \phi$. Let $z\ \varepsilon\, C\ (X, (u_n)_{n\,\varepsilon\,N})$ be fixed. By lemma 1 (iii) we get $R(\ (u_n)_{n\,\varepsilon\,N}\ )\,f(z)\ \leq \alpha(z)\,R(\ (u_n)_{n\,\varepsilon\,N}\ )(z)$. This yields – observing lemma 1 (i) and

$[\,X\,] \subset\ X - R\,(\ (u_n)_{n\,\varepsilon\,N}\ )(z) = 0$. Hence

$R\,(\ (u_n)_{n\,\varepsilon\,N}\ )\ (f\ (z)) = 0$, too, and therefore f(z) = z. The remaining part of the assertion is evident.

Theorem 5:

Let $(E, \|\quad\|)$ be a Banach space which is center – point complete (i.e., for every bounded sequence $(u_n)_{n\,\varepsilon\,N}\,\varepsilon\, E^{N}$ we have $C\,(E, (u_n)_{n\,\varepsilon\,N}) \neq \phi$ and f : E → E be a generalizer contraction. Then there is a unique x ε E such that f(x) =x and for every y ε E we have

$\lim\ \big((f^{n}(y))_{n\,\varepsilon\,N}\big) = x$ (strongly).

PROOF: Let y ε E. Since $(f^{n}(y))_{n\,\varepsilon\,N}$ is bounded by theorem 3 we have $C\ \big(E, (f^{n}(y))_{n\,\varepsilon\,N}\big) \neq \phi$. Fix $x\,\varepsilon\, C\ \big(E, (f^{n}(y)_{n\,\varepsilon\,N}\big)$. Using lemma 1 (iii) we get

$$R(\ (f^{n}(y))_{n\,\varepsilon\,N}\ )(x)\ \leq R\big((f^{n}\ (y))_{n\,\varepsilon\,N}\ \big)\ (f\,(x))$$

$$\leq \alpha(x) R(\ (f^{n}(y))_{n\,\varepsilon\,N}\ )(x).$$

Hence $R(\ (f^{n}(y))_{n\,\varepsilon\,N}\ )(x) = 0$

i.e. $\lim\ \big((f^{n}(y))_{n\,\varepsilon\,N}\big) = x$ (strongly)

and f(x) =x.

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