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Status Sum Adjacency Energy of Graphs

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ABSTRACT

The status $\sigma(u)$ of a point u in a connected graph G is the sum of the distances between u and all other vertices in G. The status sum adjacency matrix of a graph G is defined as $S_A(G) = [s_{ij}]$, where $s_{ij} = \sigma(v_i) + \sigma(v_j)$, if v_i is adjacent to v_j and $s_{ij} = 0$, otherwise. The status sum adjacency energy is defined as the sum of the absolute values of the eigenvalues of $S_A(G)$. In this paper we obtain bounds for the status sum adjacency energy of a graph.

1. Introduction

Let G be a simple, connected graph with n vertices and m edges. Let $V(G)$ be the vertex set of G and $E(G)$ be an edge set of G. The edge joining the vertices u and v is denoted by uv. The degree of a vertex u in G is the number of edges adjacent to it and is denoted by $d(u)$. The distance between two vertices u and v, denoted by $d(u,v)$ is the length of shortest path joining them. The diameter of a graph G, denoted by diam(G) is the maximum distance between any pair of vertices of G. For graph theoretic terminology we refer the book [1].

The adjacency matrix of a graph G is an $n\times n$ matrix $A(G) = [a_{ij}]$, in which $a_{ij} = 1$ if the vertices v_i and v_j are adjacent and $a_{ij} = 0$, otherwise.

The status of a vertex u in a connected graph G is defined as [2]

$$\sigma(u) = \sum_{v\in V(G)} d(u,v) \quad (1)$$

where $d(u, v)$ is the distance between the vertices u and v in G and $V(G)$ is the vertex set of G.

The status sum adjacency matrix of a connected graph G is defined as [3] $S_A(G) = [s_{ij}]$, where $s_{ij} = \sigma(v_i) + \sigma(v_j)$ if v_i and v_j are adjacent and $s_{ij} = 0$, otherwise. Let the eigenvalues of $S_A(G)$ be denoted by $x_1, x_2, \ldots, x_n$. As $S_A(G)$ is a real symmetric matrix, its eigenvalues are real.

The first status connectivity index $S_1(G)$ and second status connectivity index $S_2(G)$ of a connected graph G are defined as [2]

$$S_1(G) = \sum_{uv\in E(G)} [\sigma(u)+\sigma(v)] \quad (2)$$

and

$$S_2(G) = \sum_{uv\in E(G)} \sigma(u)\sigma(v) \quad (3)$$

where $E(G)$ is an edge set of G.

The first and second Zagreb indices of G are defined as [4]

$$Z_1(G) = \sum_{u\in V(G)} d(u)^2 = \sum_{uv\in E(G)} [d(u)+d(v)] \quad (4)$$

and

$$Z_2(G) = \sum_{uv\in E(G)} d(u)d(v) \quad (5)$$

where $d(u)$ is the degree of a vertex u in G.

The forgotten index of G is defined as [4, 5]

$$F(G) = \sum_{u\in V(G)} d(u)^3 = \sum_{uv\in E(G)} [d(u)^2+d(v)^2]. \quad (6)$$

If diameter of G is diam(G) ≤ 2, then [2]

$$S_1(G) = 4m(n-1) - Z_1(G) \quad (7)$$

and

$$S_2(G) = 4m(n-1)^2 - 2(n-1)Z_1(G) + Z_2(G). \quad (8)$$

2. On eigenvalues of $S_A(G)$

Lemma 2.1. Let G be a connected graph with n vertices and m edges. Then the eigenvalues $x_1, x_2, \ldots, x_n$ of $S_A(G)$ satisfies

$$\sum_{i=1}^{n} x_i = 0 \quad (9)$$

and

$$\sum_{i=1}^{n} x_i^2 = 2\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2 = 2M. \quad (10)$$

Proof: $\sum_{i=1}^{n} x_i = \mathrm{trace}[S_A(G)] = \sum_{i=1}^{n} s_{ii} = 0.$

$$\sum_{i=1}^{n} x_i^2 = \mathrm{trace}[S_A(G)^2] = 2\sum_{uv\in E(G)} [\sigma(u)+\sigma(v)]^2.$$

Lemma 2.2. Let G be a connected graph with n vertices and m edges. Let $x_1, x_2, \ldots x_n$ be the eigenvalues of $S_A(G)$. If diam(G) ≤ 2, then

$$\sum_{i=1}^{n} x_i = 0 \quad (11)$$

and

$$\sum_{i=1}^{n} x_i^2 = 16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G). \quad (12)$$

Proof: If diam(G) ≤ 2, then for any vertex u of G,

$\sigma(u) = 2n - 2 - d(u)$.

By Lemma 2.1,

$$\sum_{i=1}^{n} x_i = 0$$

and

$$\sum_{i=1}^{n} x_i^2 = 2\sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2$$
$$= 2\sum_{uv\in E(G)}[4n-4-d(u)-d(v)]^2$$

$$= 2\sum_{uv\in E(G)}[(4n-4)^2 - 2(4n-4)(d(u)+d(v)) + (d(u)+d(v))^2]$$

$$= 16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G).$$

Lemma 2.3. If (a_1, a_2, ... , a_n) and (b_1, b_2, …, b_n) are n-vectors, then Cauchy-Schwartz inequality is

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right). \qquad (13)$$

1. Bounds for energy of status sum adjacency energy

The status sum adjacency energy $SE_A(G)$ of a connected graph G is defined as the sum of the absolute values of the eigenvalues of $S_A(G)$. That is if x_1, x_2, …, x_n are the eigenvalues of $S_A(G)$, then

$$SE_A(G) = \sum_{i=1}^{n} |x_i|. \qquad (14)$$

The Eq. (14) is analogous to the ordinary graph energy defined as the sum of the absolute values of the eigenvalues of the adjacency matrix of G [6, 7].

Theorem 3.1. Let G be a connected graph with n vertices. Then

$$\sqrt{2M} \le SE_A(G) \le \sqrt{2nM},$$

where

$$M = \sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2.$$

Proof. Upper bound: Choosing $a_i = 1$ and $b_i = |x_i|$ for i = 1, 2, … , n in Lemma 2.3 we get,

$$\left(\sum_{i=1}^{n}|x_i|\right)^2 \le n\left(\sum_{i=1}^{n}|x_i|^2\right) = 2nM$$

$$(SE_A(G))^2 \le 2nM$$

$$SE_A(G) \le \sqrt{2nM}.$$

Lowe bound:

$$(SE_A(G))^2 = \left(\sum_{i=1}^{n}|x_i|\right)^2 \ge \sum_{i=1}^{n} x_i^2 = 2M.$$

Therefore

$$SE_A(G) \ge \sqrt{2M}.$$

Corollary 3.2. Let G be a graph with n vertices and m edges. If diam(G) ≤ 2, then

$$SE_A(G) \ge \sqrt{16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G)}$$

and

$$SE_A(G) \le \sqrt{n[16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G)]}.$$

Proof . If diam(G) ≤ 2, then by Lemma 2.2,

$$2M = \sum_{i=1}^{n} x_i^2 = 16(n-1)[2m(n-1) - Z_1(G)] + 2F(G) + 4Z_2(G).$$

Therefore by Theorem 3.1, result follows.

Theorem 3.3. Let G be a connected graph with n vertices. Then

$$SE_A(G) \ge \sqrt{2M + n(n-1)|\det(S_A(G))|^{2/n}}$$

where

$$M = \sum_{uv\in E(G)}[\sigma(u)+\sigma(v)]^2.$$

Proof. We follow the same procedure used in [8].
Consider

$$(SE_A(G))^2 = \left(\sum_{i=1}^{n}|x_i|\right)^2$$
$$= \sum_{i=1}^{n} x_i^2 + 2\sum_{1\le i<j\le n}|x_i||x_j|$$
$$= 2M + \sum_{i\ne j}|x_i||x_j|. \qquad (15)$$

As the arithmetic mean of a set of positive number is greater than or equal to their geometric mean, we have

$$\frac{1}{n(n-1)}\sum_{i\ne j}|x_i||x_j| \ge \left[\prod_{i\ne j}|x_i||x_j|\right]^{1/n(n-1)}$$

$$= \left[\prod_{i=1}^{n}|x_i|^{2(n-1)}\right]^{1/n(n-1)}$$

$$= \left[\prod_{i=1}^{n}|x_i|\right]^{2/n} = |\det(S_A(G))|^{2/n}. \qquad (16)$$

From Eqs. (15) and (16) we get,

$$SE_A(G) \ge \sqrt{2M + n(n-1)|\det(S_A(G))|^{2/n}}.$$

3. Conclusion

In this work we have obtained some properties of the eigenvalues of the status sum adjacency matrix and bounds for the status sum adjacency energy.

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