Theorems and Properties of Sadik Transform

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ABSTRACT

In the present paper, the author has established two theorems namely Initial-Value theorem and Final-Value theorem of Sadik transform and verification of these theorems are given. Sadik transform of Bessel function and it's deductions are also given.

1. Introduction

Definitions

(i) Laplace Transform

Let $f(x)$ be a real or complex valued function defined for $x > 0$, then the Laplace transform of $f(x)$, denoted by $L\{f(x);p\}$ or $F(p)$ or $\overline{F}(p)$ is defined as

$$L\{f(x);p\} = \overline{F}(p) = \int_0^\infty e^{-px} f(x)dx = \lim_{T\to\infty}\int_0^T e^{-px} f(x)dx \qquad (1.1)$$

Provided that the limit exists and finite.

(ii) Sectionally (or piecewise) Continuous Function

Let $f(x)$ be a function defined in a certain interval $[a,b]$. Suppose that $[a,b]$ can be subdivided into finite numbers of intervals such that $f(x)$ is continuous in each of these open subintervals and at the end of these subintervals the right-hand and left-hand limits exist and finite, then $f(x)$ is known as a sectionally or piecewise continuous function.

(iii) Functions of Exponential Order

A function $f(x)$ is said to be of exponential order $\gamma(\gamma > 0)$ as $x \to \infty$ if $\lim_{x\to\infty} e^{-\gamma x} f(x) =$ a finite quantity.

(iv)Functions of Class A

A function $f(x)$ is said to be of class A if

(i) It is sectionally continuous in every finite interval (for $x \geq 0$)
(ii) It is of exponential order .

(v) Libnitz's rule for differentiating under integral sign

Let $f(x,t)$ and $\frac{\partial f}{\partial x}$ are continuous functions of both variables x and t and let the first order derivatives of $g(x)$ and $h(x)$ are continuous, then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(x,t)dt = \int_{g(x)}^{h(x)} \frac{\partial f}{\partial x}dt + f(x,h(x))\frac{dh}{dx} - f(x,g(x))\frac{dg}{dx} \qquad (1.2)$$

(vi) Sumudu Transform

If $f(t) \in \left\{f(t) : M, k_1, k_2 > 0, |\, f(t)| < M.\exp\left(t/k\right)\right\}$

Then the Sumudu transform of $f(t)$ is defined by

$$F(u)=S\left[f(t)\right]=\int_0^\infty e^{-t}f(u,t)dt \quad (1.3)$$

Where the integral of R.H.S. of (1.3) is convergent.

(vii) Sadik Transform

Sadikali Latif Shaikh[1] has been defined Sadik transform in the following manner:

If $f(t)$ is piecewise continuous on the interval $0\le t\le A$ for any $A>0$ and $|f(t)|\le K.e^{w^a t}$ when $t\ge M$, for any real constant w^a and some positive constant K. Then the Sadik transform of $f(t)$ is given by

$$F\left(v^\alpha,\beta\right)=S\left[f(t)\right]=\frac{1}{v^\alpha}\int_0^\infty e^{-v^\alpha t}f(t)dt$$

, for $\operatorname{Re}\left(v^\alpha\right)>w^a$ (1.4)

Where v is complex variable, α is any non-zero real numbers and β is any real number.

For $\alpha=1,\beta=0$ the Sadik transform reduces to the Laplace transform and for $\alpha=-1,\beta=1$ it, the Sadik transform reduces to the Sumudu transform.

Some known results of Sadik transform[3]:

(i) If $f(t)=t^n$, then Sadik transform of $f(t)=t^n$ is

$$S\left[t^n\right]=F\left(v^\alpha,\beta\right)=\frac{n!}{v^{(n+1)\alpha+\beta}} \quad (1.5)$$

(ii) $$S\left[\sin at\right]=F\left(v^\alpha,\beta\right)=\frac{av^{-\beta}}{v^{2\alpha}+a^2} \quad (1.6)$$

(iii) $$S\left[\cos at\right]=F\left(v^\alpha,\beta\right)=\frac{v^{\alpha-\beta}}{v^{2\alpha}+a^2} \quad (1.7)$$

(iv) $$S\left[\sinh at\right]=F\left(v^\alpha,\beta\right)=\frac{av^{-\beta}}{v^{2\alpha}-a^2} \quad (1.8)$$

(v) $$S\left[\cosh at\right]=F\left(v^\alpha,\beta\right)=\frac{v^{\alpha-\beta}}{v^{2\alpha}-a^2} \quad (1.9)$$

(vi) $$S\left[e^{at}\right]=F\left(v^\alpha,\beta\right)=\frac{v^{-\beta}}{v^\alpha-a} \quad (1.10)$$

(vii) If $G\left[x,v^\alpha,\beta\right]$ is a Sadik transform of $\varphi(x,t)$ and $\varphi_t(x,t)$ is a first order partial derivative of $\varphi(x,t)$ with respected to t, then

$$S\left[\varphi_t(x,t)\right]=v^\alpha G\left(x,v^\alpha,\beta\right)-v^{-\beta}\varphi(x,0) \quad (1.11)$$

(viii) If $G\left[x,v^\alpha,\beta\right]$ is a Sadik transform of $\varphi(x,t)$ and $\varphi_{tt}(x,t)$ is a second order partial derivative of $\varphi(x,t)$ with respected to t, then

$$S\left[\varphi_{tt}(x,t)\right]=v^{2\alpha}G\left(x,v^\alpha,\beta\right)-v^{\alpha-\beta}\varphi(x,0)-v^{-\beta}\varphi_t(x,0) \quad (1.12)$$

2. Theorems

In this section, we will establish Initial-Value theorem and Final-Value theorem of Sadik transform and verification of these theorems are also given.

Theorem1. Initial-Value Theorem

Let $f(t)$ be continuous for all $t\ge 0$ and be of exponential order as $t\to\infty$. Also suppose that $f^{'}(t)$ is of class A, then

$$\lim_{t\to 0}f(t)=v^\beta\lim_{v^\alpha\to\infty}v^\alpha S\left\{f(t)\right\} \quad (2.1)$$

Proof: By the result (1.11), we have

$$S\left\{f^{'}(t)\right\}=v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0)$$

Or $$\frac{1}{v^{\beta}}\int_0^{\infty}e^{-v^{\alpha}t}f^{'}(t)dt=v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0) \qquad (2.2)$$

But if $f^{'}(t)$ is sectionally continuous and of exponential order, we have

$\lim\limits_{v^{\alpha}\to\infty}\int_0^{\infty}e^{-v^{\alpha}t}f^{'}(t)dt=0$. Taking limit as $v^{\alpha}\to\infty$ in (2.2), we find that

$$0=\lim_{v^{\alpha}\to\infty}v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0)$$

Or $$f(0)=v^{\beta}\lim_{v^{\alpha}\to\infty}v^{\alpha}S\left\{f(t)\right\} \qquad (2.3)$$

Since $f(t)$ is continuous at $t\to 0$, we have

$f(0)=\lim\limits_{t\to 0}f(t)$. Thus (2.3) gives

$$\lim_{t\to 0}f(0)=v^{\beta}\lim_{v^{\alpha}\to\infty}v^{\alpha}S\left\{f(t)\right\}$$

Theorem2. Final-Value Theorem

Let $f(t)$ be continuous for all $t\geq 0$ and be of exponential order as $t\to\infty$. Also suppose that $f^{'}(t)$ is of class A, then

$$\lim_{t\to\infty}f(t)=v^{\beta}\lim_{v^{\alpha}\to 0}v^{\alpha}S\left\{f(t)\right\} \qquad (2.4)$$

Proof: By the result (1.11), we have

$$S\left\{f^{'}(t)\right\}=v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0)$$

Or $$\frac{1}{v^{\beta}}\int_0^{\infty}e^{-v^{\alpha}t}f^{'}(t)dt=v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0) \qquad (2.5)$$

The limit of the L.H.S. of (2.5) as $v^{\alpha}\to\infty$ is

$$\lim_{v^{\alpha}\to\infty}\frac{1}{v^{\beta}}\int_0^{\infty}e^{-v^{\alpha}t}f^{'}(t)dt=\frac{1}{v^{\beta}}\int_0^{\infty}f^{'}(t)dt=$$

$$\lim_{T\to\infty}\frac{1}{v^{\beta}}\int_0^{T}f^{'}(t)dt=\lim_{T\to\infty}\left\{\frac{f(t)-f(0)}{v^{\beta}}\right\}=\lim_{T\to\infty}\frac{f(t)}{v^{\beta}}-\frac{f(0)}{v^{\beta}}$$

The limit of the R.H.S. of (2.5) as $v^{\alpha}\to 0$ is

$$\lim_{v^{\alpha}\to 0}v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0).$$

Thus $$\lim_{t\to\infty}\frac{f(t)}{v^{\beta}}-v^{-\beta}f(0)=\lim_{v^{\alpha}\to 0}v^{\alpha}S\left\{f(t)\right\}-v^{-\beta}f(0)$$

Or $$\lim_{t\to\infty}f(t)=v^{\beta}\lim_{v^{\alpha}\to 0}v^{\alpha}S\left\{f(t)\right\}$$

Verification of Initial-Valur theorem and Final-Value theorem

Let $f(t)=e^{-2t}$, then $S\left\{e^{-2t}\right\}=\frac{v^{-\beta}}{v^{\alpha}+2}$ (by using (1.10))

The Initial-Value theorem is

$$\lim_{t\to 0}f(t)=v^{\beta}\lim_{v^{\alpha}\to\infty}v^{\alpha}S\left\{f(t)\right\}$$

Here $\lim\limits_{t\to 0}e^{-2t}=1$ and $v^{\beta}\lim\limits_{v^{\alpha}\to\infty}v^{\alpha}S\left\{f(t)\right\}=v^{\beta}\lim\limits_{v^{\alpha}\to\infty}\frac{v^{-\beta}v^{\alpha}}{v^{\alpha}+2}=1$

Hence the Initial-Value theorem is verified.

The Final-Value theorem is

$$\lim_{t\to\infty} f(t) = v^{\beta} \lim_{v^{\alpha}\to 0} v^{\alpha} S\{f(t)\}$$

Here $\lim_{t\to\infty} e^{-2t} = 0$ and

$$v^{\beta} \lim_{v^{\alpha}\to 0} v^{\alpha} S\{f(t)\} = v^{\beta} \lim_{v^{\alpha}\to 0} \frac{v^{-\beta} v^{\alpha}}{v^{\alpha}+2} = 0$$.

Hence the Final-Value theorem is verified.

Sadik Transform of Bessel function

For $\mathrm{Re}(v^{\alpha}) > a > 0$

$$S\left\{t^n J_n(at)\right\} = \frac{(2a)^n \Gamma(n+\frac{1}{2})}{v^{\beta}\sqrt{\pi}\left(v^{2\alpha}+a^2\right)^{n+\frac{1}{2}}}$$ for $n > \frac{1}{2}$ (2.6)

Proof:

$$S\left\{t^n J_n(at)\right\} = S\left\{t^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{at}{2}\right)^{n+2r}\right\}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{a}{2}\right)^{n+2r} S\left\{t^{2n+2r}\right\} = \sum_{r=0}^{\infty} \frac{(-1)^r \Gamma(2n+2r+1)}{r!\Gamma(n+r+1) v^{(2n+2r+1)\alpha+\beta}} \left(\frac{a}{2}\right)^{n+2r}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r 2^{2n+2r} \pi^{-\frac{1}{2}} \Gamma\left(n+r+\frac{1}{2}\right)\Gamma(n+r+1)}{r!\Gamma(n+r+1) v^{(2n+2r+1)\alpha+\beta}} \left(\frac{a}{2}\right)^{n+2r}$$

[By applying Gamma duplication formula]

$$= \frac{(2a)^n \Gamma(n+\frac{1}{2})}{v^{(2n+1)\alpha+\beta}\sqrt{\pi}} \sum_{r=0}^{\infty} \frac{\left(n+\frac{1}{2}\right)_r}{r!} \left(-\frac{a^2}{v^{2\alpha}}\right)^r$$

$$= \frac{(2a)^n \Gamma(n+\frac{1}{2})}{v^{(2n+1)\alpha+\beta}\sqrt{\pi}} \left(1+\frac{a^2}{v^{2\alpha}}\right)^{-n-\frac{1}{2}} \quad [\because \sum_{r=0}^{\infty} \frac{n_r(-a)^r}{r!} = (1+a)^r]$$

Deduction:

(i) Putting $n = 0$ in (2.6), we have

$$S\left\{J_0(at)\right\} = \frac{1}{v^{\beta}\sqrt{v^{2\alpha}+a^2}}$$ (2.7)

(ii) $$S\left\{tJ_0(at)\right\} = -\frac{d}{dv^{\alpha}} S\left\{J_0(at)\right\} = \frac{1}{v^{\beta}(v^{2\alpha}+a^2)^{\frac{3}{2}}}$$ (2.8)

(iii) $$S\left\{e^{-at} J_0(bt)\right\} = \frac{1}{v^{\beta}\sqrt{(v^{\alpha}+a)^2+b^2}}$$ (2.9)

(iv) $$\int_0^{\infty} J_0(t)dt = v^{-\beta}$$ (2.10)

Proof: $\int_0^\infty e^{-v^\alpha t} J_0(t)dt = \frac{1}{v^\beta \sqrt{v^{2\alpha}+1}}$. Taking limit as $v^\alpha \to 0$, we arrive at the result.

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