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Partial Differential Equations and Application to Mathematical

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ABSTRACT

Partial differential equations in organize flow simulations bring an extra arrangement of mathematical and numerical issues. For a distributed model to be mathematically very much postured, appropriate initial and boundary conditions must be determined. Boundary condition prerequisites for nonlinear unit models may change over the span of a dynamic simulation, even without discrete occasions. Some distributed models, because of shameful plan or basic interpretation blunders, might be not well postured in light of the fact that they don't have a mathematical property called continuous dependence on data.

1. Introduction

A standout amongst the most noteworthy Indian commitments to the theory of elliptic partial differential equations is crafted by S Minakshisundaram. His work brought about the beginning of a prolific research region, referred to today as geometric unearthly asymptotic. Geometric otherworldly asymptotics is the investigation of the connection between the coarse components of the geometry of a space and those of the spectrum of the Laplace operator. There was a surge of advancements here in the 1970s. Together with A Pleijel(1949)1, he presented a capacity known as the Minakshisundaram– Pleijel zeta work, comparable to the renowned Riemann zeta work. The buildups at the poles of this capacity give information about the arrived at the midpoint of thickness of the eigenvalues in the high recurrence constrain and furthermore on Eigen functions.

To simply refer to one application of this work, we review the acclaimed question of Marc Kac, 'Would one be able to hear the state of a drum?' In mathematical terms, this means the question whether two spaces which have a similar spectrum for the Laplace operator (they are then called isospectral areas) are compatible to each other. This problem, postured around 1966, was settled, contrarily, in the 1990s. In any case, utilizing crafted by Minakshisundaram and Pleijel, done in the 1950s, it can be demonstrated that the spectrum settles the border of a two-dimensional space. That it additionally settles the volume, originates from a praised work of Weyl done in the early piece of the twentieth century. In this way, utilizing the classical isoperimetric inequality, we quickly observe that, on the off chance that one of two isospectral areas is a plate, so is the other. Accordingly, we can "hear" the state of a roundabout drum!

A partial differential equation (PDE) is an equation stating a relationship between a function of two or more independent variables and the partial derivatives of this function with respect to these independent variables. The dependent variable f is used as a generic dependent variable. In most problems in engineering and science, the independent variables are either space (x, y, z) or space and time (x, y, z, t). The dependentvariable depends on the physical problem being modeled. Examples of three simple partial differential equations having two independent variables are presented below:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \quad (1)$$

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \quad (2)$$

$$\frac{\partial^2 f}{\partial t^2} = c^2 \frac{\partial^2 f}{\partial x^2} \quad (3)$$

Equation (1) is the two-dimensional Laplace equation, Eq. (2) is the one-dimensional diffusion equation, and Eq. (3) is the one-dimensional wave equation. For simplicity of notation, Eqs. (1) to (3) usually will be written as

$$f_{xx} + f_{yy} = 0 \quad (4)$$

$$f_t = \alpha f_{xx} \quad (5)$$

$$f_{tt} = c^2 f_{xx} \quad (6)$$

where the subscripts denote partial differentiation.

[1]Minakshisundaram S and Pleijel A 1949 Some properties of the Eigen functions of the Laplace operator on Riemannian manifolds; Canadian J. Math. **1** 242–256.

The solution of a partial differential equation is that particular function, $f(x, y)$ or $f(x, t)$, which satisfies the PDE in the domain of interest, $D(x, y)$ or $D(x, t)$, respectively, and satisfies the initial and/or boundary conditions specified on the boundaries of the domain of interest. In a very few special cases, the solution of a PDE can be expressed in closed form. In the majority of problems in engineering and science, the solution must be obtained by numerical methods.

Equations (4) to (6) are examples of partial differential equations in two independent variables, x and y, or x and t. Equation (4), which is the two-dimensional Laplace equation, in three independent variables is

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = 0 \quad (7)$$

where ∇^2 is the Laplacian operator, which in Cartesian coordinates is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad (8)$$

Equation (5), which is the one-dimensional diffusion equation, in four independent variables is

$$f_t = \alpha(f_{xx} + f_{yy} + f_{zz}) = \alpha \nabla^2 f \quad 9)$$

The parameter α is the diffusion coefficient. Equation (6), which is the one-dimensional wave equation, in four independent variables is

$$f_{tt} = c^2(f_{xx} + f_{yy} + f_{zz}) = c^2 \nabla^2 f \quad (10)$$

The parameter c is the wave propagation speed. Problems in two, three, and four independent variables occur throughout engineering and science.2

Equations (III.4) to (III. 10) are all second-order partial differential equations. The order of a PDE is determined by the highest-order derivative appearing in the equation. A large number of physical problems are governed by second-order PDEs. Some physical problems are governed by a first-order PDE of the form

$$af_t + bf_x = 0 \quad (11)$$

where a and b are constants. Other physical problems are governed by fourth-order PDEs such as

$$f_{xxxx} + f_{xxyy} + f_{yyyy} = 0 \quad (12)$$

Equations (4) to (12) are all linear partial differential equations. A linear PDE is one in which all of the partial derivatives appear in linear form and none of the coefficients depends on the dependent variable. The coefficients may be functions of the independent variables, in which case the PDE is a linear, variable coefficient, PDE. For example,

$$af_t + bxf_x = 0 \quad (13)$$

where a and b are constants, is a variable coefficient linear PDE, whereas Eqs. (4) to (12) are all linear PDEs. If the coefficients depend, on the dependent variable, or the derivatives appear in a nonlinear form, then the PDE is nonlinear. For example,

$$ff_x + bf_y = 0 \quad (14)$$

$$af_x^2 + bf_y = 0 \quad (15)$$

are nonlinear PDEs.

Equations (4) to (15) are all homogeneous partial differential equations. An example of a nonhomogeneous PDE is given by

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = F(x, y, z) \quad (16)$$

Equation (16) is the nonhomogeneous Laplace equation, which is known as the Poisson equation. The nonhomogeneous term, $F(x, y, z)$, is a forcing function, a source term, or adissipation function, depending on the application. The appearance of a nonhomogeneous term in a partial differential equation does not change the general features of the PDE, nor does it usually change or complicate the numerical method of solution.

Equations (4) to (16) are all examples of a single partial differential equation governing one dependent variable. Many physical problems are governed by a system of PDEs involving several dependent variables. For example, the two PDEs

$$af_t + bg_x = 0 \quad (17a)$$

$$Ag_t + Bf_x = 0 \quad (17b)$$

[2]Fritz John, Partial Di_erential Equations (3th Edn), Applied Mathematical Sciences 1, Springer-Verlag, Heidelberg-Berlin-New York, 2001.

comprise a system of two coupled partial differential equations in two independent variables (x and t) for determining the two dependent variables, $f(x,t)$ and $g(x,t)$. Systems containing several PDEs occur frequently, and systems containing higher-order PDEs occur occasionally. Systems of PDEs are generally more difficult to solve numerically than a single PDE.3

The deduction of the equations of motion for a perfect uid by Euler in 1755, and after that for a gooey uid by Navier (1822) and Stokes (1845) were a visit de-power of eighteenth and nineteenth century science. These equations have been utilized to depict and clarify such a large number of physical wonders around us in nature, that as of now billions of dollars of research concedes in arithmetic, science and engineering now rotate around them. They can be utilized to model the coupled environmental and sea flow utilized by the meteorological office for climate forecast down to any application in chemical engineering you can consider, say to improvement of the thrusters on NASA's Apollo program rockets. The incompressible Navier{Stokes equations are given by

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nu \nabla^2 \boldsymbol{u} - \frac{1}{\rho}\nabla p + \boldsymbol{f},$$
$$\nabla \cdot \boldsymbol{u} = 0,$$

where $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},t)$ is a three-dimensional fluid velocity, $p = p(\boldsymbol{x},t)$ is the pressureandf is an external force field. The constants ρ and ν are the mass density and kinematic viscosity, respectively. The frictional force due to stickiness of a fluid is represented by the term $\nu \nabla^2 \boldsymbol{u}$. An ideal fluid corresponds to the case $\nu = 0$, when the equations aboveare known as the Euler equations for a homogeneous incompressible ideal fluid. We will derive the Navier-Stokes equations and in the process learn about the subtleties of fluid mechanics and along the way see lots of interesting applications.

A material displays flow if shear powers, however little, prompt a misshaping which is unbounded we could utilize this as definition of a fluid. A strong has a settled shape, or if nothing else a solid restriction on its misshaping when constrain is connected to it. With the classification of fluids", we incorporate fluids and gasses. The principle recognizing highlight between these two fluids are the thought of compressibility. Gasses are normally compressible as we know from ordinary pressurized canned products and air canisters. Fluids are by and large incompressible an include basic to all present day auto braking components.

Fluids can be further subcategorized. There are perfect or inviscid Fluids. In such fluids, the main inside power display is weight which acts with the goal that fluid flows from a district of high weight to one of low weight. The equations for a perfect fluid have been connected to wing and airplane design (as a farthest point of high Reynolds number flow). However, fluids can show inward frictional powers which model a \stickiness" property of the fluid which includes vitality loss these are known as thick Fluids. A few fluids/materials known as non-Newtonian or complex fluids" display considerably more bizarre conduct, their response to Misshaping may rely upon: (i) previous history (prior distortions), for instance a few paints; (ii) temperature, for instance a few polymers or glass; (iii) the measure of the disfigurement, for instance a few plastics or silly putty.4

2. The Vibrating String

Our next goal is to derive the equation which governs the motion of a vibrating string. We consider a string of length *L* stretched out along the x-axis, one end of the string being at $x = 0$ and the other being at $x = L$. We assume that the string is free to move only in the vertical direction. Let $u(x, t)$ = vertical displacement of the string at the point x at time *t*.

We will derive a partial differential equation for $u(x,t)$. Note that since the ends of the string are fixed, we must have $u(0,t) = 0 = u(L,t)$ for all *t*.

It will be convenient to use the "configuration space" V_0. An element $u(x) \in V_0$ represents a configuration of the string at someinstant of time. We will assume that the *potential energy* in the string when it is in the configuration $u(x)$ is

$$V(u(x)) = \int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx,$$

where *T* is a constant, called the *tension* of the string.

Indeed, we could imagine that we have devised an experiment that measures the potential energy in the string in various configurations, and has determined that does indeed represent the total potential energy in the string. On the other hand, this expression for potential energy is quite plausible for the following reason: We could imagine first that the amount of energy in the string should be proportional to the amount of stretching of the string, or in other words, proportional to the length of the string.

[3]Je_rey Rauch, Partial Di_erential Equations, Graduate Text in Mathematics 128, Springer-verlag, Heidelberg-Berlin-New York, 2001.
[4]Chorin, A.J. and Marsden, J.E. 1990 A mathematical introducton to uid mechanics, Third edition, Springer{Verlag, New York.

From vector calculus, we know that the length of the curve $u = u(x)$ is given by the formula Length $\int_0^L \sqrt{1+(du/dx)^2}dx.$

But when du/dx is small,

$$\left[1+\frac{1}{2}\left(\frac{du}{dx}\right)^2\right]^2 = 1+\left(\frac{du}{dx}\right)^2 + \text{a small error}$$

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and hence

$\sqrt{1+(du/dx)^2}$ is closely approximated by $1+\frac{1}{2}(du/dx)^2$

Thus to a first order of approximation, the amount of energy in the string should be proportional to

$$\int_0^L \left[1+\frac{1}{2}\left(\frac{du}{dx}\right)^2\right]dx = \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}.$$

Letting T denote the constant of proportionality yields

Energy in string = $\int_0^L \frac{T}{2}\left(\frac{du}{dx}\right)^2 dx + \text{constant}$

Potential energy is only defined up to addition of a constant, so we can drop the constant term to obtain.

The force acting on a portion of the string when it is in the configuration $u(x)$ is determined by an element $F(x)$ of V_0. We imagine that the force acting 011 the portion of the string from x to $x + dx$ is $F(x)dx$. When the forcepushes the string through an infinitesimal displacement $\xi(x) \in V_0$, the total*work* performed by $F(x)$ is then the "sum" of the forces acting on the tiny pieces of the string, in other words, the work is the "inner product" of F and ξ,

$$\langle F(x), \xi(x)\rangle = \int_0^L F(x)\xi(x)dx$$

On the other hand, this work is the amount of potential energy lost when the string undergoes the displacement:

$$\langle F(x), \xi(x)\rangle = \int_0^L \frac{T}{2}\left(\frac{\partial u}{\partial x}\right)^2 dx - \int_0^L \frac{T}{2}\left(\frac{\partial(u+\xi)}{\partial x}\right)^2 dx$$

$$= -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx + \int_0^L \frac{T}{2}\left(\frac{\partial \xi}{\partial x}\right)^2 dx.$$

We are imagining that the displacement ξ is infinitesimally small, so terms containing the square of ξ or the square of a derivative of ξ can be ignored, and hence

$$\langle F(x), \xi(x)\rangle = -T\int_0^L \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial x}dx$$

Integration by parts yields

$$\langle F(x), \xi(x)\rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx - T\left(\frac{\partial u}{\partial x}\xi\right)(L) - T\left(\frac{\partial u}{\partial x}\xi\right)(0)$$

Since. $\xi(0) = \xi(L) = 0$ we conclude that

$$\int_0^L F(x)\xi(x)dx = \langle F(x), \xi(x)\rangle = T\int_0^L \frac{\partial^2 u}{\partial x^2}\xi(x)dx$$

Since this formula holds for *all* infinitesimal displacements $\xi(x)$, we must have

$$F(x) = T\frac{\partial^2 u}{\partial x^2}$$

for the force density per unit length.

Now we apply Newton's second law, force = mass x acceleration, to the function u(x.t). The force acting 011 a tiny piece of the string of length dx is F(x)dx, while the mass of this piece of string is just ρdx, where ρ is the density of the string. Thus Newton's law becomes

$$T\frac{\partial^2 u}{\partial x^2}dx = \rho dx\frac{\partial^2 u}{\partial t^2}$$

If we divide by ρdx, we obtain the wave equation,

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

where $c^2 = T/\rho$

Just as in the preceding section, we could approximate this partial differential equation by a system of ordinary differential equations. Assume that the string has length *L* = 1 and set $x_i = i/n$ and

$u_i(t) = u(x_i, t)$ = the displacement of the string at x_i at time *t* .

Then the function *u (x , t*) can be approximated by the vector-valued function

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \cdot \\ u_{n-1}(t) \end{pmatrix}$$

of one variable, just as before. The wave equation is then approximated by the system of ordinary differential equations

$$\frac{d^2\mathbf{u}}{dt^2} = c^2n^2P\mathbf{u},$$

where *P* is the (n — 1) x (n — 1) matrix described in the preceding section. Thus the differential operator

$\mathbf{L} = \frac{d^2}{dx^2}$ is approximated by the symmetric matrix n^2P,

and we expect solutions to the wave equation to behave like solutions to a mechanical system of weights and springs with a large number of degrees of freedom.[5]

3. Wave Equation

The motivation behind this section is to think about initial-boundary esteem problems for the wave equation in one space dimension. Specifically, we will infer formal arrangements by a partition of factors method, build up uniqueness of the arrangement by vitality contentions, and study properties of limited contrast approximations.

The wave equation models the development of a versatile, homogeneous string which experiences moderately little transverse vibrations. The wave equation is of second order regarding the space variable x and time t, and takes the frame

$$u_{tt} = c^2u_{xx}. \quad (18)$$

Here the constant c is called the wave speed. Since the equation is of second order with respect to time, an initial value problem typically needs two initial conditions. Hence, in addition to the differential equation (18) we specify two initial conditions of the form

$$u(x,0) = f(x) \quad \text{and} \quad u_t(x,0) = g(x). \quad (19)$$

If we study the pure initial value problem, i.e. where *x* varies over all of $\mathbb{R}$, then the solution of (18)-(19) is given by d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)dy; \quad (20)$$

Throughout this chapter we shall consider the following initial and boundary value problem:

[5] P.-L. Chow, Stochastic partial differential equations, Chapman & Hall/CRC Applied Mathematics and Nonlinear Science Series. Boca Raton, FL: Chapman & Hall/CRC. ix, 281 p., 2007.

$$u_{tt} = u_{xx} \quad \text{for} \quad x \in (0,1)\,,\ t > 0,$$
$$u(0,t) = u(1,t) = 0, \quad t > 0,$$
$$u(x,0) = f(x)\,, \quad u_t(x,0) = g(x)\,,\ x \in (0,1) \tag{21}$$

We note that we have assumed that the wave speed c is set equal to 1. In fact, any problem with $c \neq 0$ can be transformed to a problem with *c* = 1
by introducing a proper time scale. Therefore, we set *c* = 1 for simplicity.

Separation of Variables - Let us try to find solutions of problem (21) of the form $u(x,t) = X(x)T(t).$
By inserting this ansatz into the wave equation, we obtain

$$X(x)T''(t) = X''(x)T(t)$$

or

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} \tag{22}$$

We can argue that since the left-hand side is independent of *x* and the right-hand side is independent of t, both expressions must be independent of *x* and *t*. Therefore,

$$\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda \tag{23}$$

for a suitable $\lambda \in \mathbb{R}$. In particular this means that the functions *X(x)* satisfy the eigenvalue problem

$$-X''(x) = \lambda X(x), \quad x \in (0,1),$$
$$X(0) = X(1) = 0, \tag{24}$$

where the boundary conditions follow from (21). Of course, this eigenvalue problem is by now familiar to us. Then we conclude that

$$\lambda = \lambda_k = (k\pi)^2 \text{ for } k = 1, 2, \ldots \tag{25}$$

with corresponding Eigenfunctions

$$X_k(x) = \sin(k\pi x) \text{ for } k = 1, 2, \ldots\ . \tag{26}$$

On the other hand, the functions $T_k(t)$ must satisfy

$$-T_k''(t) = \lambda_k T_k(t) = (k\pi)^2 T_k(t).$$

This equation has two linearly independent solutions given by

$$T_k(t) = e^{ik\pi t} \text{ and } T_k(t) = e^{-ik\pi t}. \tag{27}$$

The general *real* solution is therefore of the form

$$T_k(t) = a_k \cos(k\pi t) + b_k \sin(k\pi t),$$

where a_k, $b_k \in \mathbb{R}$ are arbitrary constants. Hence, we conclude that the functions

$$u_k(x,t) = \sin(k\pi x)\big(a_k \cos(k\pi t) + b_k \sin(k\pi t)\big) \tag{28}$$

satisfy the differential equation and the boundary values prescribed by the initial-boundary value problem. [6]Furthermore, these solutions satisfy the initial conditions

$$u_k(x,0) = a_k \sin(k\pi x) \text{ and } (u_k)_t(x,0) = b_k k\pi \sin(k\pi x)$$

In order to obtain more solutions, we can add solutions of the form (9.11) and obtain

$$u(x,t) = \sum_{k=1}^{N} \sin(k\pi x)\big(a_k \cos(k\pi t) + b_k \sin(k\pi t)\big) \tag{29}$$

with initial conditions

[6] C. Großmann and H.-G. Roos, Numerics of partial differential equations. (NumerikpartiellerDifferentialgleichungen.), TeubnerStudienbucher: Mathematik. Stuttgart: B.G. Teubner. 477 p., 2002.

$$u(x,0) = \sum_{k=1}^{N} a_k \sin(k\pi x) \quad \text{and} \quad u_t(x,0) = \sum_{k=1}^{N} b_k k\pi \sin(k\pi x). \tag{30}$$

EXAMPLE Consider the problem with $f(x) = 2\sin(\pi x)$ and $g(x) = -\sin(2\pi x)$. Hence, the initial data is of the form (30) with

$a_1 = 2$, $a_k = 0$ for $k > 1$

and

$b_2 = -\frac{1}{2\pi}$, $b_k = 0$ for $k \neq 2$

The solution *u*(*x*, *t*) is therefore given by

$$u(x,t) = 2\sin(\pi x)\cos(\pi t) - \frac{1}{2\pi}\sin(2\pi x)\sin(2\pi t)$$

This solution is plotted in Fig. 1.

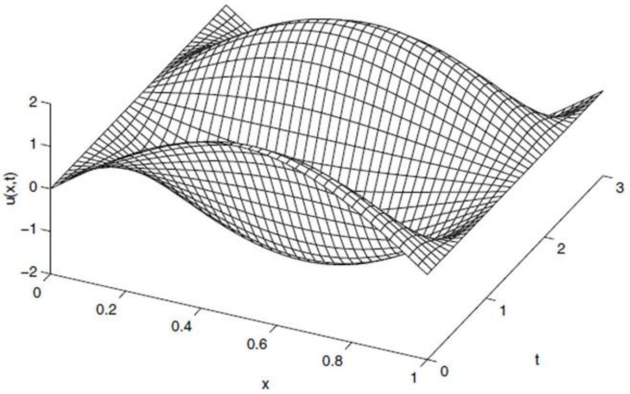

Figure 1. The solution u(x, t) derived in Example for $(x,t) \in ([0,1]\times[0,3])$

In order to cover a larger class of initial functions, we allow general Fourier sine series as initial functions, i.e. we let *N* tend to infinity in (9.12) and (9.13). Hence, if

$$f(x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x) \quad \text{and} \quad g(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x) \tag{31}$$

then we obtain a formal solution of the initial-boundary value problem given by

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left(a_k\cos(k\pi t) + \frac{b_k}{k\pi}\sin(k\pi t)\right) \tag{32}$$

EXAMPLE Consider the initial-boundary value problem with

$f(x) = x(1-x)$ and $g(x) = 0$

The Fourier sine series of *f* is given by

$$f(x) = \sum_{k=1}^{\infty} \frac{8}{\pi^3(2k-1)^3}\sin((2k-1)\pi x)$$

Hence, by (9.4) the formal solution is given by

$$u(x,t) = \sum_{k=1}^{\infty} \frac{8}{\pi^3(2k-1)^3}\sin((2k-1)\pi x)\cos((2k-1)\pi t)$$

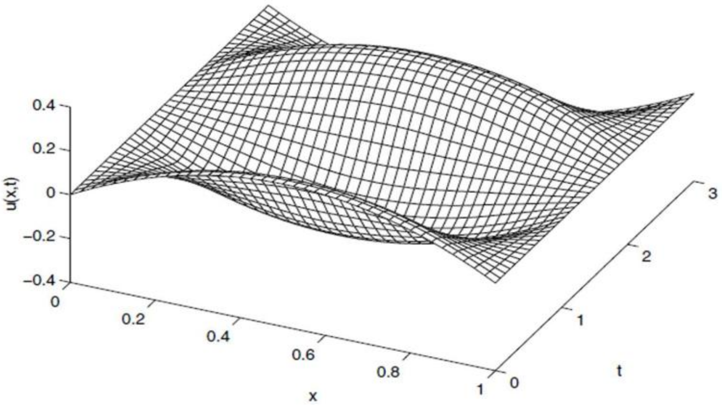

Figure 2. The solution u (x, t) derived in Example for $(x,t) \in ([0,1]\times[0,3])$ using the first 20 terms of the series.

In Fig. 2 we have plotted this formal solution by using the first 20 terms of this infinite series.

4. Conclusion

Differential equations are partitioned into two classes, ordinary and partial. An ordinary differential equation (ODE) includes one independent variable and derivatives as for that variable. A partial differential equation (PDE) includes more than one independent variable and relating partial derivatives. The short survey of new methods of factorization, automatization and correct linearization of the ordinary differential equations is spoken to. These methods alongside the strategy for the gathering investigation in light of utilizing both point and nonpoint, nearby and nonlocal changes are viable apparatuses for investigation of nonlinear self-ruling and no autonomous dynamical frameworks. Therefore, an extent of precisely resolvable problems of the Nonlinear examination is broadened.

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