Volume-04
Issue-05
May-2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com[UGC Listed Journal]

# The Continuants of Transcendental Numbers in Continued Fraction Expansion

Dr. Shashi Bhushan Rai

*Associate Professor, Department of Mathematics, B.N.College Patna , Bihar (India)*

**ARTICLE DETAILS**

**Article History**
*Published Online:* 25 May 2019

**Keywords**
*Transcendental numbers, Continued fraction, Hausdorff dimension, maximal.*

**ABSTRACT**

In this paper, we want to explain that if a purely transcendental set is determined merely by the properties of the individual continuants, besides algebraic numbers, most transcendental numbers are excluded from this set.
Namely, let φ be a positive function defined on N and set

$A\ (\varphi) = \{\ x \in [\ :1,0\ )\ q_n\ (x\ ) \geq \varphi\ (\ n\ )$ , infinitely many $n$ }.

If $A\ (\varphi\ )$ is a purely transcendental set, then the set $A\ (\varphi\ )$ is of Hausdorff dimension at most one-half.

Let $q_n(\ \theta\ )$ be $n$-th continuant of θ in its continued fraction expansion. For any irrational θ ∈ [0,1) .
Davenport and Roth showed that if θ satisfies

$loglog\, q_n > \frac{cn}{\sqrt{logn}}$ for, infinitely many $n$ ∈ N , for all $c$ > ,0 then θ must be transcendental.
We say a set $A$ purely transcendental set, if all the elements in $A$ are transcendental.

## 1. Introduction

Diophantine approximation is intimated connected with continued fractions in the sense that, for any irrational θ ∈ [ 0,1) if

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{2q^2},$$

then $\frac{p}{q}$must be a convergent of θ in its continued fraction expansion.

It seems that the first result concerning the properties of continuants of transcendental number or equivalently algebraic numbers is Liouville inequality [1], which shows that any algebraic number of degree $d$ cannot be
approximated by rational numbers at an order greater than $d$. Using this result, one has

**Theorem 1.1** ([1]). Let θ be an irrational and q n(θ ) be the n-th continuants in its continued fraction expansion. If, for any c

$$loglog\, q_n(\ \theta\ ) \geq cn \text{ for infinitely many n} \in \text{N} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.1)$$

then θ is transcendental.

According to an estimation on the number of solutions to the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{2q^{2+\delta}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1.2)$$

when ξ is algebraic, Davenport and Roth [2] derived an improvement of (1.1). **Theorem 1.2** ([2]). Let θ be an irrational and $q_n(\theta)$ be the n-th continuants in its continued fraction
expansion. If for any c > 0

$$loglog\, q_n > \frac{cn}{\sqrt{logn}} \text{for, infinitely many } n \in \text{N} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1.3)$$

then θ is transcendental.

An qualitative improvement of the number of solutions to inequality

(1.2), which is given by Bombieri and van der Poorten [6], enable Adamczewski and Bugeaud [3] to obtain an improved one.

**Theorem 1.3** ([3])**.** Let θ be an irrational and $\mathrm{q_n}(\theta)$ be the n-th continuants in its continued fraction expansion. If for any 0 > c,

$$\mathrm{loglogq_n} > \mathrm{cn^{\frac{2}{3}}(logn)^{\frac{2}{3}}} \quad \text{for infinitely many } n \in N \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(1.4)}$$

then θ must be transcendental.

We say a set A purely transcendental set, if all the elements in A are transcendental. In this paper, we intend to explain that if a purely transcendental set is characterized merely by precise properties of the individual continuants, besides algebraic numbers, most transcendental numbers are also excluded from this set.

Namely, let φ be a positive function defined on N and set

$$\mathrm{A}(\varphi) = \{\mathrm{x} \in [\,0{,}1): \mathrm{q_n(x)} \geq \varphi(\mathrm{n}),\ infinitely\ \ many\ n\,\},$$

*We show that*

**Theorem 1.4.** If A( φ ) is a purely transcendental set, then the set A( φ) is of Hausdorff dimension at most one-half.

## 2. Preliminaries

We begin with some notations firstly. Let x∈ [0,1 ) be an irrational number and$[a_1(x), a_2(x), \ldots\ldots\ldots]$be its regular continued fraction expansion. For any n ≥1 denote by

$$\mathrm{p_n(x)/q_n(x)} := [a_1(x), a_2(x), \ldots\ldots\ldots]$$

the *n*-th convergent of *x*. With the conventions that

$p_{-1}(x) = 1,\ q_{-1}(x) = 0,\ \ p_0(x) = 0, q_0(x) = 1,$ one has [4]

p n + 1 (x ) =a n + 1 (x ) pn (x ) + pn − 1 (x ) , n≥ 0

qn + 1 ( x) = an + 1 (x ) n (x ) + qn − 1 (x ) ,n ≥ 0 .......................(2.1)

where { *q* n }≥1 are commonly called the *continuants* .

For any a1, a2................a n∈ N , denote by I n ( a1 , a2.......,a n ) the n-th cylinder

I n (a1 , a2.......... a n ) = { x ∈ [0,1 ) : a 1 (x ) =a 1a 2......... , , a n (x ) =a n }.

**Lemma 2.1 ([4])**. For any a 1 ,a 2............a n∈ N ,

$$|I_n(a_{1\,,}a_{2\,,\ldots\ldots\ldots}a_{n\,,})| = \frac{1}{q_n(q_n + q_{n+1})}, \prod_{j=1}^{n} a_j < q_n < \prod_{j=1}^{n}(a_j + 1)$$

where|. | denotes the length of a subset in [0,1 ) and $q_n, q_{n+1}$are recursively defined by (2.1).
Let φ be a positive function on N Set

E ( φ ) = { x ∈ [ 0,1 ): a n (x ) ≥ φ ( n ) for infinitely many n }.
A complete result on the Hausdorff dimension of E (φ ) was given in [5], but only the needed part is cited here.

**Lemma 2.2** ([5])**.** *Write* $b = exp\left\{\lim_{n\to\infty} inf\ \frac{loglog\ \emptyset(n)}{n}\right\}$. *If*

$$\lim_{n\to\infty} inf\ \frac{log\emptyset(n)}{n} = \infty$$

*then* $dim_H E(\emptyset) = \frac{1}{b+1}$

Recall that
*A* (φ) = { *x* ∈ [ 0,1 ): *q n* ( x) ≥ φ ( *n* ) for infinitely many *n* }.

**Lemma 2.3.** *If A* (φ ) *is a purely transcendental set*, *then*

$$\lim_{n\to\infty} \inf \frac{\log\emptyset(n)}{n} = \infty$$

**Proof:**- For any integer *B* ≥ 1 let *xB* = [ *B,B*,........, ]

Lagrange's theorem asserts that *x B*is quadratic irrational. Whence,

$\varphi(n) \geq q_n(x_B) \geq B^n$ for *n* ultimately .

## 3. Proof of Main Result

In this section, we give the exact Hausdorff dimension of the purely transcendental set A (φ).

**Lemma 3.1.** *If A* (φ) *is a purely transcendental set*, *then*

$$\dim_H A(\varphi) = \frac{1}{b+1} \text{ where } b = exp\left\{\lim_{n\to\infty} inf \frac{loglog\emptyset(n)}{n}\right\}$$

**Proof**:-According to Lemma 2.3, the lower bound of dim *H A* (φ) is a direct consequence of Lemma 2.2.

Now we turn to the upper bound. Two cases will be distinguished according as *b* = 1 or *b* > 1
(i)If *b* = 1, for any *t* > 1, we introduce a family of measures

$u_t : u_t\big(I_n(a_1, a_2, \ldots\ldots a_n)\big) = e^{-np(t)-t\sum_{j=1}^{n} log\, a_j(x)}$ ..................... 3.1

where $p(t) = log \sum_{n=1}^{\infty} \frac{1}{n^t}$

Now, let I (n ) be the family of all *n*-th order cylinders$I(a_1, a_2, \ldots\ldots a_n)$

which satisfies *qn* ≥ φ ( *n* ). Then,

$$A(\varphi) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{I_n(a_1,a_2,\ldots\ldots a_n)\in I(n)} I_n(a_1, a_2, \ldots\ldots a_n)$$

For each *N* ≥ 1 we select all those cylinders in $\bigcup_{n=N}^{\infty} I(n)$
which are maximal
maximal($I \in \bigcup_{n=N}^{\infty} I(n)$)is maximal if there is no other *I'* in
such that I ⊂ I ' and I≠ I' ).
We denote by J ( N) the set of all maximal cylinders in $\bigcup_{n=N}^{\infty} I(n)$

It is evident that J (N ) is a cover of A (φ) for any N ≥ 1
Fix t $> 1$ and $\varepsilon > 0$. Choose N0large enough such that for any

$n > N_{0\,,}\ \varepsilon log\varphi(n) \geq np(t)$.

Fix$N > N_{0\,,}$ .Then for any$I_n(a_1, a_2, \ldots\ldots a_n) \in J(N)$, we have

$$|I_n(a_1, a_2, \ldots\ldots a_n)|^{\frac{i+\varepsilon}{2}} \leq e^{-(i+\epsilon) log\, q n} \leq e^{-t\sum_{j=1}^{n} log\ aj - np\ (t)} = \mu_t\big(I_n(a_1, a_2, \ldots\ldots a_n)\big) \leq 1$$

This implies $\dim A(\varphi) \leq \frac{1}{2} = \frac{1}{b+1}$

(ii) If $b > 1$ ,By the definition of b , one has, for any $\epsilon > 0$,

$\prod_{j=1}^{n} e^{(b-\epsilon)^{j-1}}(b-\epsilon-1) \leq \varphi(\mathrm{n})$,for n ultimately. .......... 3.2

So, it follows

$$A(\varphi) \in \{x \in [\,0,1): \mathrm{a_n(x)} \geq [e^{(b-\epsilon)^{n-1}}(b-\epsilon-1)] - 1, \mathrm{i.0.}, \mathrm{n}\}$$

As a consequence of Lemma 2.2, one gets $\dim_{\mathrm{H}} \mathrm{A}(\varphi) \leq \frac{1}{\mathrm{b}+1}$

## References

[1] Y. Bugeaud, Approximation by Algebraic Numbers, Cambridge Tracts Math. 160, Cambridge, 2004.

[2] H. Davenport and K. F. Roth, Rational approximations to algebraic numbers, Mathematika 2 (1955), 160-167.

[3] B. Adamczewski and Y. Bugeaud, On the Maillet-Baker continued fractions, J. Reine Angew. Math. 606 (2007), 105-121.

[4] A. Ya. Khintchine, Continued Fractions, P. Noordhoff, Groningen, The Netherlands, 1963.

[5] B. W. Wang and J. Wu, Hausdorff, dimension of certain sets arising in continued fraction expansions, Adv. Math. 218 (2008), 1319-1339.

[6] E. Bombieri and A. J. van der Poorten, Some quantitative results related to Roth's theorem, J. Austral. Math. Soc. Ser. A 45 (1988), 233-248.