RESEARCH REVIEW JOURNALS
Volume-04
Issue-04
April -2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Best Proximity Point Theorems for m- Rational Proximal Contraction Mappings in Metric Spaces.

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ARTICLE DETAILS

Article History
Published Online: 15 April 2019

Keywords
Proximal contraction , Best Proximity Point , Converges , fixed point, complete metric space, Uniqueness .

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ABSTRACT

In this paper, we define a new class of best proximity point theorems for m-rational proximal contraction in the metric spaces, and also proved the uniqueness and convergence of best proximity point theorems for rational proximal m- contraction of the first and second kind in complete metric spaces. Various examples are illustrated in this paper.

INTRODUCTION AND PRELIMINARIES

In a mathematical point of view ,there are quite number of problems arising in the form of areas of natural science involving the existence of solutions of nonlinear equations with the form $Tx = x$,Where T is a selfmapping or non-self mapping defined on metric spaces or topological vector spaces. If T is not a selfmapping , then probably we can say the fixed point equation $Tx = x$ does not possess a solution then the normal intrigue is to discover an element $x \in X$ such that x is in proximity to Tx in some sense. In other words we would like to get a desirable estimate for the quantity d(x,Tx) is minimum. So the idea of best proximity point was discussed [8], The most elegant and beautiful theorem in this field was the Banach's contraction mapping principle [6].In the last decades fixed point and Best proximity point were investigated by several authors [2,3, 6, 1, 4,5] . For two subsets A and B of X and T is a non-self mapping from A to B, a best proximity pair theorem investigates the conditions affirming the presence of a component x such that

d (x,Tx) = dist (A,B).

Definition 1.1 Let X be a nonempty set . A function $d : X \times X \to R^{+}$ is called a metric provide that ,for all $x, y, z \in X$,

1. $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$
3. $d(x, z) \le d(x, y) + d(y, z)$.

A pair (X,d) is called a metric space.

Definition 1.2: Let (X,d) be a metric space $\{x_n\}$ be a sequence in X and $x \in X$. Then,

a) The sequence $\{x_n\}$ is said to be convergent in (X,d) and converges to x ,if for every $\varepsilon > 0$ there exists a $n_0 \in N$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$ and this fact is represented by $\lim\limits_{n\to\infty} = x$ or $x_n \to x$ as $n \to \infty$.

b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X,d) if for every $\varepsilon > 0$ there exits $n_0 \in N$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n < n_0, p > 0$ or equivalently ,if $\lim\limits_{n\to\infty}(x_n, x_{n+p}) = 0$ for all p

c) (X,d) is said to be a complete metric space if every Cauchy sequence in X converges to some $x \in X$.

Defintion1.3: A subset k of a metric space X is boundedly compact if each bounded sequence in k has subsequence converging to a point in k . suppose X is a uniformly convex (and hence reflexive) Banach space with modules convexity δ .Then $\delta(\varepsilon) > 0 \ for \varepsilon > 0$ and $\delta(.)$ is strictly increasing. Moreover ,if $x, y, p \in X, R > 0$ and $r \in [0, 2R]$,

$$\left.\begin{array}{l}\|x-p\| \le R\\ \|y-p\| \le R\\ \|x-y \ge R\end{array}\right\} \Rightarrow \left\|\frac{x+y}{2} - p\right\| \le \left(1 - \delta\left(\frac{r}{R}\right)\right)R$$

Definition 1.4: The set B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ of B satisfying the condition that $d(x, y_n) \to d(x, B)$ for some $x \in A$ has a convergent subsequence.

In this setting , we recall the following notions:

$$dist(A,B) = \inf\{d(a,b) / a \in A, b \in B\}$$

$$A_0 = \{a \in A / d(a,b) = dist(A,B) \; for some \; b \in B\}$$

$$B_0 = \{b \in B / d(a,b) = dist(A,B) \; for some \; a \in A\}$$

MAIN RESULTS

Definition 2.1:

Let (X,d) be a complete metric spaces . A mapping $T : A \to B$ is said to be m –rational proximal contraction of the first kind if there exist non-negative real numbers t_1, t_2, t_3, t_4, t_5 with $t_1 + t_2 + t_3 + t_4 + 2t_5 < 1$ such that the conditions $d(u_1, Tv_1) = d(u_2, Tv_2) = dist(A,B)$ imply that

$$d(u_1,u_2) \le t_1 d(v_1,v_2) + t_2 \frac{1+d(v_1,u_1)}{1+d(v_1,v_2)} d(v_2,u_2)$$
$$+t_3 \frac{[1+d(v_1,u_1)]d(v_2,u_2) + d(v_2,u_2)d(v_2,u_1)}{1+d(v_1,v_2)} +$$
$$t_4 d(v_1,u_1) + t_5[d(v_1,u_2) + d(v_2,u_1)]$$

for all $u_1, u_2, v_1, v_2 \in A$.

Definition 2.2: Let (X,d) be a complete metric spaces. A mapping $T : A \to B$ is said to be m –rational proximal contraction of the second kind if there exist non-negative real numbers t_1, t_2, t_3, t_4, t_5 with $t_1 + t_2 + t_3 + t_4 + 2t_5 < 1$ such that the condition $d(u_1, Tv_1) = d(u_2, Tv_2) = dist(A,B)$ imply that

$$d(Tu_1,Tu_2) \le t_1 d(Tv_1,Tv_2) + t_2 \frac{1+d(Tv_1,Tu_1)}{1+d(Tv_1,Tv_2)} d(Tv_2,Tu_2)$$
$$+t_3 \frac{[1+d(Tv_1,Tu_1)]d(Tv_2,Tu_2) + d(Tv_2,Tu_2)d(Tv_2,Tu_1)}{1+d(Tv_1,Tv_2)}$$
$$+t_4 d(Tv_1,Tu_1) + t_5\left[d(Tv_1,Tu_2) + d(Tv_2,Tu_1)\right]$$

for all $u_1, u_2, v_1, v_2 \in A$.

Theorem 2.3: Let (X,d) be a complete metric space. Let A and B be non-empty, closed subsets of X such that B is approximately compact with respect to A. Suppose that A_0 and B_0 be non-empty and $T : A \to B$ be a non-self mapping satisfying the following conditions.

a) T is a m – rational proximal contraction of the first kind.

b) $T(A_0) \subseteq B_0$

Then, there exist a unique element $v \in A$ such that $d(v,Tv) = dist(A,B)$. Then for each $v_0 \in A_0$, the sequence $\{v_n\}$ define by $d(v_{n+1}, Tv_n) = dist(A,B)$, converges to the best proximity point v.

Proof : Let $v_0 \in A_0$. Since B_0 contains $T(A_0)$, so there exist $v_1 \in A_0$ such that $d(v_1, Tv_0) = dist(A,B)$.

Since $Tv_1 \in B_0 \supseteq T(A_0)$. Again there exist $v_2 \in A_0$ such that $d(v_2, Tv_1) = dist(A,B)$.

Thus we obtain a sequence $\{v_n\} \in A_0$, there exist an element $v_{n+1} \in A_0$ such that $d(v_{n+1}, Tv_n) = dist(A,B)$ for every $n \in \mathbb{N} \cup \{0\}$.

.

By definition (2.1), we have

$$d(v_n, v_{n+1}) \le t_1 d(v_{n-1}, v_n) +$$
$$t_2 \frac{1+d(v_{n-1},v_n)}{1+d(v_{n-1},v_n)} d(v_n,v_{n+1})$$
$$+t_3 \frac{[1+d(v_{n-1},v_n)]d(v_n,v_{n+1}) + d(v_n,v_{n+1})d(v_n,v_n)}{1+d(v_{n-1},v_n)}$$
$$+t_4 d(v_{n-1},v_n) + t_5\left[d(v_{n-1},v_{n+1}) + d(v_n,v_n)\right]$$
$$\le t_1 d(v_{n-1},v_n) + t_2 d(v_n,v_{n+1})$$
$$+t_3 d(v_n,v_{n+1}) + t_4 d(v_{n-1},v_n)$$
$$+t_5[d(v_{n-1},v_n) + d(v_n,v_{n+1})]$$
$$d(v_n,v_{n+1}) \le \frac{t_1+t_4+t_5}{1-(t_2+t_3+t_5)} d(v_{n-1},v_n)$$

$d(v_n,v_{n+1}) \le \beta d(v_{n-1},v_n)$,where

$$\beta = \frac{t_1+t_4+t_5}{1-(t_2+t_3+t_5)}.$$

Repeating this process, we get

$$d(v_n,v_{n+1}) \le \beta^n d(v_0,v_1)$$

For any m,n and $m > n$, we have

$$d(v_n,v_m) \le d(v_n,v_{n+1}) + d(v_{n+1},v_m)$$
$$\le d(v_n,v_{n+1}) + d(v_{n+1},v_{n+2}) +$$
$$d(v_{n+2},v_m) + \ldots + d(v_{m-1},v_m)$$
$$= \beta^n\left[1+\beta+\beta^2+\ldots+(\beta)^{m-n-1}\right]d(v_0,v_1)$$

For $0 \le \beta < 1$ we have

$$d(v_n,v_m) = \frac{\beta^n}{1-\beta} d(v_0,v_1)$$

Taking $m,n \to \infty$

$$\lim_{m,n\to\infty} d(v_n,v_m) \to 0$$

Hence $\{v_n\}$ is a Cauchy sequence in X. Then there exists $v \in A$ such that the sequence $\{v_n\} \to v$. Thus

$d(v,B) \le d(v,Tv_n) = \lim_{n\to\infty} d(v_{n+1},Tv_n) = dist(A,B) \le d(v,B).$

By Definition (2.1), we have

$d(v,Tv) = dist(A,B) = d(v_{n+1},Tv_n)$,we get

$$d(x,v_{n+1}) \le t_1 d(v,v_n) + t_2 \frac{1+d(v,x)}{1+d(v_n,v)} d(v_n,v_{n+1})$$

$$+t_3 \frac{1+d(v,x)d(v_n,v_{n+1})+d(v_n,v_{n+1})d(v_n,x)}{1+d(v,v_n)}$$

$$+t_4 d(v,x) + t_5\left[d(v,v_{n+1})+d(v_n,x)\right]$$

Letting $n \to \infty$

$$d(x,v) \le (t_4+t_5) d(x,v)$$

This implies, $v = x$, since $t_4 + t_5 < 1.$

Therefore $d(v,Tv) = d(x,Tv) = dist(A,B)$

Hence T has a best proximity point $v \in A.$

Next we prove the uniqueness of the best proximity point.

Let w be the another best proximity point of T, so that $d(w,Tw) = dist(A,B)$

By Definition (2.1), we have

$$d(v,w) \le t_1 d(v,w) + t_2 \frac{1+d(v,v)}{1+d(v,w)} d(w,w)$$

$$+t_3 \frac{[1+d(v,v)]d(w,w)+d(w,w)d(w,v)}{1+d(v,w)}$$

$$+l_4 d(v,v) + l_5\left[d(v,w)+d(w,v)\right]$$

$$d(v,w) \le (t_1+2t_5) d(v,w)$$

It follows that $v = w$ since $t_1 + 2t_5 < 1.$ Hence T has a unique best proximity point.

Theorem 2.4 : Let (X,d) be a complete metric space . Let A and B be non-empty, closed subsets of X such that A is approximately compact with respect to $B.$ Suppose that A_0 and B_0 be non-empty and $T: A \to B$ be a non-self maping satisfying the following conditions.

a)T is continuous m- rational proximal contraction of the second kind.

b) $T(A_0) \subseteq B_0$

Then, there exist a unique element $v \in A$ such that $d(v,Tv) = dist(A,B)$.Then for each $v_0 \in A_0$,the sequence $\{v_n\}$ define by $d(v_{n+1},Tv_n) = dist(A,B),$ converges to the best proximity point v .

Proof: As in the proof of Theorem 2.3 , we can find a sequence $\{v_n\}$ in A_0 such that $d(v_{n+1},Tv_n) = dist(A,B)$ for all non-negative integer $n,$ since T is a m-rational proximal contraction of the second kind, we get

$$d(Tv_n,Tv_{n+1}) \le t_1 d(Tv_{n-1},Tv_n) +$$

$$t_2 \frac{1+d(Tv_{n-1},Tv_n)}{1+d(Tv_{n-1},Tv_n)} d(Tv_n,Tv_{n+1})$$

$$+t_3 \frac{[1+d(Tv_{n-1},Tv_n)]d(Tv_n,Tv_{n+1})+d(Tv_n,Tv_{n+1})d(Tv_n,Tv_n)}{1+d(Tv_{n-1},Tv_n)}$$

$$+t_4 d(Tv_{n-1},Tv_n) + t_5\left[d(Tv_{n-1},Tv_{n+1})+d(Tv_n,Tv_n)\right]$$

$$\le t_1 d(Tv_{n-1},Tv_n) + t_2 d(Tv_n,Tv_{n+1})$$

$$+t_3 d(Tv_n,Tv_{n+1}) + t_4 d(Tv_{n-1},Tv_n)$$

$$+t_5[d(Tv_{n-1},Tv_n)+d(Tv_n,Tv_{n+1})]$$

$$d(Tv_n,Tv_{n+1}) \le \frac{t_1+t_4+t_5}{1-(t_2+t_3+t_5)} d(Tv_{n-1},Tv_n)$$

$d(Tv_n,Tv_{n+1}) \le \beta d(Tv_{n-1},Tv_n)$,where

$$\beta = \frac{t_1+t_4+t_5}{1-(t_2+t_3+t_5)}.$$

Continuing this process, we obtain

$$d(Tv_n,Tv_{n+1}) \le \beta^n d(Tv_0,Tv_1)$$

For any m,n and $m > n$ we obtain

$$d(Tv_n,Tv_m) \le d(Tv_n,Tv_{n+1}) + d(Tv_{n+1},Tv_m)$$

$$d(Tv_n,Tv_m) \le d(Tv_n,Tv_{n+1}) + d(Tv_{n+1},Tv_{n+2}) +$$

$$d(Tv_{n+2},Tv_m) + \dots + d(Tv_{m-1},Tv_m)$$

$$= \beta^n\left[1+\beta+\beta^2+\dots+(\beta)^{m-n-1}\right] d(Tv_0,Tv_1)$$

For $0 \le \beta < 1$ we have

$$d(Tv_n,Tv_m) = \frac{\beta^n}{1-\beta} d(Tv_0,Tv_1)$$

Taking $m,n \to \infty$

$$\lim_{m,n\to\infty} d(Tv_n,Tv_m) \to 0$$

Hence $\{Tv_n\}$ is a Cauchy sequence in X. Since (X,d) is complete and B is closed the sequence $\{Tv_n\}$ converges to some $y \in B.$

$$d(y,A) \le d(y,v_{n+1})$$

$$\le d(y,Tv_n) + d(Tv_n,v_{n+1})$$

$$\le d(y,Tv_n) + dist(A,B)$$

$$= d(y,Tv_n) + d(y,A)$$

Letting $n \to \infty$, we obtain

$$d(y,A) \le \lim_{n\to\infty} d(y,z_{n+1}) \le d(y,A)$$

Thus $d(y, v_n) \to d(y, A)$. Since A is approximately compact with respect to B, the sequence $\{v_n\}$ has a subsequence $\{Tv_{n_k}\}$ converges to some element $v \in A$. By continuity of T, we have

$$d(v, Tv) = \lim_{k\to\infty} d(v_{n_k+1}, Tv_{n_k}) = dist(A, B)$$

Therefore, v is a proximity point of T.
Next we prove the uniqueness of the best proximity point.
Let z be the another best proximity point of T,

$$d(z, Tz) = dist(A, B)$$

$$d(Tv, Tz) = t_1 d(Tv, Tz) +$$

$$t_2 \frac{1 + d(Tv, Tv)}{1 + d(Tv, Tz)} d(Tz, Tz)$$

$$+t_3 \frac{[1 + d(Tv, Tv)]d(Tz, Tz) + d(Tz, Tz)d(Tz, Tv)}{1 + d(Tv, Tz)}$$

$$+t_4 d(Tv, Tv) + t_5[d(Tv, Tz) + d(Tv, Tz)]$$

we get,

$$d(Tv, Tz) \le (t_1 + 2t_5) d(Tv, Tz)$$

It follows that $Tv = Tz$ since $t_1 + 2t_5 < 1$. Hence T has a unique best proximity point.
This completes the proof.

Theorem 2.5 : Let A and B be non-empty, closed subsets of a complete metric space (X, d). Suppose that A_0 and B_0 are non empty and non-empty and $T : A \to B$ is a mapping satisfying the following conditions

a) T is the m-rational proximal contraction of the first as well as m- rational proximal contraction of the second kind.

b) $T(A_0) \subseteq B_0$.

Then there exists unique element $v \in A$ such that $d(v, Tv) = dist(A, B)$ and the sequence $\{v_n\}$ Converges to the best proximity point v where v_0 is any fixed element in A and $d(v_{n+1}, Tv_n) = dist(A, B)$ for all $n \ge 0$.

Proof : Proceeding as in the proof of the theorem 2.3, we find a sequence $\{v_n\}$ in A_0 such that

$$d(v_{n+1}, Tv_n) = dist(A, B)$$

for all non-negative integer n.

By theorem 2.3, we can prove that the sequence $\{v_n\}$ is a Cauchy sequence and hence every sequence converges to some $v \in A$.

As in Theorem 2.4, it can be proved that the sequence $\{Tv_n\}$ is a Cauchy sequence and hence every converges to some $y \in B$. So, we get

Therefore $v \in A_0$. Since $T(A_0) \subseteq B_0$, we have $d(u, Tv) = dist(A, B)$ for some $u \in A$.

Since T is a m-rational proximal contraction of the first kind we obtain.

$$d(u, v_{n+1}) \le t_1 d(v, v_n) + t_2 \frac{1 + d(v, u)}{1 + d(v_n, v)} d(v_n, v_{n+1})$$

$$+t_3 \frac{[1 + d(v, u)]d(v_n, v_{n+1}) + d(v_n, v_{n+1})d(v_n, u)}{1 + d(v, v_n)}$$

$$+t_4 d(v, u) + t_5[d(v, v_{n+1}) + d(v_n, u)]$$

Letting $n \to \infty$

$$d(u, v) \le (t_4 + t_5) d(v, u)$$

Since $(t_4 + t_5) < 1 \Rightarrow d(u, v) = 0$ therefore $u = v$.

Thus it follows, $d(v, Tv) = d(u, Tv) = dist(A, B)$. Hence v is a best proximity point of T. We can prove that the uniqueness of the best proximity point of the mapping T as in Theorem 2.1. This completes the proof.

Example 2.4:

Let $X = \Re^2$ with the metric,

$$d((v, y), d(v^1, y^1)) = \max\{|v + v^1|, |y + y^1|\}$$

and let $A = \{(v, y) : v \in [-2, -1] \cup [1, 3], y \in [-1, 1]\}$

and $B = [-1, 1]$. Then d(A,B) = 1. Where A_0 = A and B_0 = B ,
Define a mapping T: A $\to$ B by

$$T(v, y) = \begin{cases} (0, v+2), if & v \le -1, \\ (0, v-2), if & v \ge 1 \end{cases}$$

In this set $t_1 = \frac{1}{20}, t_2 = \frac{1}{10}, t_3 = \frac{1}{6}, t_4 = \frac{1}{4}, t_5 = \frac{1}{5}$,

then $t_1 + t_2 + t_3 + t_4 + 2t_5 < 1$.

Pick $u_1 = (-2, 1), u_2 = (3, -1), v_1 = (-1, 0), v_2 = (1, 0)$ then $Tu_1 = (0, 0), Tu_2 = (0, 1), Tv_1 = (0, 1), Tv_2 = (0, -1)$,

Therefore $d(v_1, Tu_1) = d(v_2, Tu_2) = dist(A, B) = 1$.

Also

$$d(v_1, v_2) = 0, d(u_1, u_2) = 1, d(v_1, u_1) = 3$$

$$d(v_1, u_2) = 2, d(v_2, u_2) = 4, d(v_2, u_1) = 1$$

we can see that,

$$d(u_1,u_2) \le t_1 d(v_1,v_2) + t_2 \frac{1+d(v_1,u_1)}{1+d(v_1,v_2)} d(v_2,u_2)$$

$$+t_4 d(v_1,u_1) + t_5[d(v_1,u_2)+d(v_2,u_1)]$$

Hence T is a m- rational proximal contraction of the first kind. Now to check the m- rational proximal contraction of the second kind,

$$d(Tv_1,Tv_2)=0, d(Tu_1,Tu_2)=1, d(Tv_1,Tu_1)=1$$
$$d(Tv_1,Tu_2)=2, d(Tv_2,Tu_2)=0, d(Tv_2,Tu_1)=1$$

we can see that,

$$d(Tu_1,Tu_2) \le t_1 d(Tv_1,Tv_2) + t_2 \frac{1+d(Tv_1,Tu_1)}{1+d(Tv_1,Tv_2)} d(Tv_2,Tu_2)$$

$$+t_3 \frac{[1+d(Tv_1,Tu_1)]d(Tv_2,Tu_2)+d(Tv_2,Tu_2)d(Tv_2,Tu_1)}{1+d(Tv_1,Tv_2)}$$

$$+t_4 d(Tv_1,Tu_1) + t_5\left[d(Tv_1,Tu_2)+d(Tv_2,Tu_1)\right]$$

Hence T is not a m- rational proximal contraction of the second kind .

Conclusion:

In this research article, we proved best proximity point theorems for m-rational proximal contraction in the setting of metric spaces. And also its an interesting paper to find the uniqueness of the best proximity point of the mapping T, where T is a m-rational proximal contraction of first as well as second kind. We presented many interesting results in this paper and these new results would attract many researchers in the recent trends.

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