

# Analytical Approach to Fixed Point Theorem on G-Metric Space Mappings

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## ABSTRACT

In this paper, the authors have studied some fixed point theorems for mapping satisfying sufficient conditions on complete G-metric space. Also they studied that if the G-metric space  $(X, G)$  is symmetric, then the existence and uniqueness of these fixed point results follow from well-known theorems in usual metric space  $(X, d_G)$ , where  $(X, d_G)$  is the usual metric space which defined from the G-metric space  $(X, G)$ .

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## Introduction

During the sixties, the notion of 2-metric space introduced by Gähler [1] as a generalization of usual notion of metric space  $(X, d)$ . But different authors proved that there is no relation between these two functions, for instance, Ha et al. in [3] show that 2-metric need not be continuous function, further there is no easy relationship between results obtained in the two settings.

In 1992, Bapure Dhage in his Ph.D. thesis introduce a new class of generalized metric space called *D*-metric spaces ([4, 5]).

In a subsequent series of papers, Dhage attempted to develop topological structures in such spaces (see [5–7]). He claimed that *D*-metrics provide a generalization of ordinary metric functions and went on to present several fixed point results.

But in 2003 in collaboration with Brailey Sims, we demonstrated in [8] that most of the claims concerning the fundamental topological structure of *D*-metric space are incorrect, so, we introduced more appropriate notion of generalized metric space as follows.

**Definition 1.1.** Let  $X$  be a nonempty set, and let  $G : X \times X \times X \rightarrow \mathbf{R}^+$  be a function satisfying the following properties:

(G1)  $G(x, y, z) = 0$  if  $x = y = z$ ;

(G2)  $0 < G(x, x, y)$ ; for all  $x, y \in X$ , with  $x \neq y$ ;

(G3)  $G(x, x, y) \leq G(x, y, z)$ , for all  $x, y, z \in X$ , with  $z \neq y$ ;

(G4)  $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ , (symmetry in all three variables);

(G5)  $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ , for all  $x, y, z, a \in X$ , (rectangle inequality).

Then the function  $G$  is called a *generalized metric*, or, more specifically, a *G-metric* on  $X$ , and the pair  $(X, G)$  is called a *G-metric space*.

**Definition 1.2** Let  $(X, G)$  be a G-metric space, and let  $(x_n)$  be sequence of points of  $X$ , a point  $x \in X$  is said to be the *limit* of the sequence  $(x_n)$ , if  $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$ , and one says that the sequence  $(x_n)$  is *G-convergent* to  $x$ .

Thus, that if  $x_n \rightarrow x$  in a G-metric space  $(X, G)$ , then for any  $\epsilon > 0$ , there exists  $N \in \mathbf{N}$  such that  $G(x, x_n, x_m) < \epsilon$ , for all  $n, m \geq N$ .

**Proposition 1.3** Let  $(X, G)$  be a G-metric space, then the following are equivalent.

(1)  $(x_n)$  is G-convergent to  $x$ .

(2)  $G(x_n, x_n, x) \rightarrow 0$ , as  $n \rightarrow \infty$ .

(3)  $G(x_n, x, x) \rightarrow 0$ , as  $n \rightarrow \infty$ .

(4)  $G(x_m, x_n, x) \rightarrow 0$ , as  $m, n \rightarrow \infty$ .

**Definition 1.4** Let  $(X, G)$  be a G-metric space, a sequence  $(x_n)$  is called *G-Cauchy* if for every  $\epsilon > 0$ , there is  $N \in \mathbf{N}$  such that  $G(x_n, x_m, x_l) < \epsilon$ , for all  $n, m, l \geq N$ ; that is, if  $G(x_n, x_m, x_l) \rightarrow 0$  as  $n, m, l \rightarrow \infty$ .

**Proposition 1.5** If  $(X, G)$  is a G-metric space, then the following are equivalent.

- (1) The sequence  $(x_n)$  is G-Cauchy.
- (2) For every  $\epsilon > 0$ , there exists  $N \in \mathbf{N}$  such that  $G(x_n, x_m, x_m) < \epsilon$ , for all  $n, m \geq N$ .

**Definition 1.6** Let  $(X, G)$  and  $(X^d, G^d)$  be two G-metric spaces, and let  $f : (X, G) \rightarrow (X^d, G^d)$  be a function, then  $f$  is said to be G-continuous at a point  $a \in X$  if and only if, given  $\epsilon > 0$ , there exists  $\delta > 0$  such that  $x, y \in X$ ; and  $G(a, x, y) < \delta$  implies  $G^d(f(a), f(x), f(y)) < \epsilon$ . A function  $f$  is G-continuous at  $X$  if and only if it is G-continuous at all  $a \in X$ .

**Proposition 1.7** Let  $(X, G), (X^d, G^d)$  be two G-metric spaces. Then a function  $f : X \rightarrow X^d$  is G-continuous at a point  $x \in X$  if and only if it is G sequentially continuous at  $x$ ; that is, whenever  $(x_n)$  is G-convergent to  $x$ ,  $(f(x_n))$  is G-convergent to  $f(x)$ .

**Definition 1.8** A G-metric space  $(X, G)$  is called symmetric G-metric space if  $G(x, y, y) = G(y, x, x)$  for all  $x, y \in X$ .

**Proposition 1.9** Let  $(X, G)$  be a G-metric space, then the function  $G(x, y, z)$  is jointly continuous in all three of its variables.

**Definition 1.10** A G-metric space  $(X, G)$  is said to be G-complete (or complete G-metric) if every G-Cauchy sequence in  $(X, G)$  is G-convergent in  $(X, G)$ .

**Proposition 1.11** A G-metric space  $(X, G)$  is G-complete if and only if  $(X, d_G)$  is a complete metric space.

**Main Results**

Here we start our work with the following theorem.

**Theorem 2.1.** Let  $(X, G)$  be a complete G-metric space, and let  $T : X \rightarrow X$  be a mapping satisfying one of the following conditions:

$$G(T(x), T(y), T(z)) \leq \{aG(x, y, z) + bG(x, T(x), T(x)) + cG(y, T(y), T(y)) + dG(z, T(z), T(z))\} \quad (2.1)$$

or

$$G(T(x), T(y), T(z)) \leq \{aG(x, y, z) + bG(x, x, T(x)) + cG(y, y, T(y)) + dG(z, z, T(z))\} \quad (2.2)$$

for all  $x, y, z \in X$  where  $0 \leq a + b + c + d < 1$ , then  $T$  has a unique fixed point (say  $u$ , i.e.,  $Tu = u$ ), and  $T$  is G-continuous at  $u$ .

**Proof.** Suppose that  $T$  satisfies condition (2.1), then for all  $x, y \in X$ , we have

$$\begin{aligned} G(Tx, Ty, Ty) &\leq aG(x, y, y) + bG(x, Tx, Tx) + (c + d)G(y, Ty, Ty), \\ G(Ty, Tx, Tx) &\leq aG(y, x, x) + bG(y, Ty, Ty) + (c + d)G(x, Tx, Tx). \end{aligned} \quad (2.3)$$

Suppose that  $(X, G)$  is symmetric, then by definition of metric  $(X, d_G)$  and (1.2), we get

$$d_G(Tx, Ty) \leq ad_G(x, y) + \frac{c + d + b}{2}d_G(x, Tx) + \frac{c + d + b}{2}d_G(y, Ty), \quad \forall x, y \in X. \quad (2.4)$$

In this line, since  $0 < a + b + c + d < 1$ , then the existence and uniqueness of the fixed point follows from well-known theorem in metric space  $(X, d_G)$  (see [10]).

However, if  $(X, G)$  is not symmetric then by definition of metric  $(X, d_G)$  and (1.3), we get

$$d_G(Tx, Ty) \leq ad_G(x, y) + \frac{2(c + d + b)}{3}d_G(x, Tx) + \frac{2(c + d + b)}{3}d_G(y, Ty), \quad (2.5)$$

for all  $x, y \in X$ , then the metric condition gives no information about this map since  $0 < a + 2(c + d + b)/3 + 2(c + d + b)/3$  need not be less than 1. But this can be proved by G-metric.

**Corollary 2.2.** Let  $(X, G)$  be a complete G-metric space and let  $T : X \rightarrow X$  be a mapping satisfying one of the following conditions:

$$\begin{aligned} G(T^m(x), T^m(y), T^m(z)) \\ \leq aG(x, y, y) + bG(x, T^m(x), T^m(x)) + cG(y, T^m(y), T^m(y)) + dG(z, T^m(z), T^m(z)) \end{aligned} \quad (2.6)$$

or

$$G(T^m(x), T^m(y), T^m(z)) \leq aG(x, y, y) + bG(x, x, T^m(x)) + cG(y, y, T^m(y)) + dG(z, z, T^m(z)) \quad 2.7$$

for all  $x, y, z \in X$ , where  $0 \leq a + b + c + d < 1$ . Then  $T$  has a unique fixed point (say  $u$ ), and  $T^m$  is  $G$ -continuous at  $u$ .

*Proof.* From the previous theorem, we see that  $T^m$  has a unique fixed point (say  $u$ ), that is,  $T^m(u) = u$ . But  $T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u))$ , so  $T(u)$  is another fixed point for  $T^m$  and by uniqueness  $Tu = u$ .

**Theorem 2.3.** Let  $(X, G)$  be a complete  $G$ -metric space, and let  $T : X \rightarrow X$  be a mapping satisfying one of the following conditions:

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, T(y), T(y)), G(y, T(x), T(x))\} \tag{2.8}$$

or

$$G(T(x), T(y), T(y)) \leq k \max\{G(x, x, T(y)), G(y, y, T(x))\}, \tag{2.9}$$

for all  $x, y \in X$ , where  $k \in [0, 1)$ , then  $T$  has a unique fixed point (say  $u$ ), and  $T$  is  $G$ -continuous at  $u$ .

*Proof.* Since whenever the mapping satisfies condition (2.8), or (2.9), then it satisfies condition (2.6), or (2.7), respectively, in Theorem 2.3. Then the proof follows from Theorem 2.5.

**Theorem 2.4.** Let  $(X, G)$  be a complete  $G$ -metric space, and let  $T : X \rightarrow X$ , be a mapping satisfying one of these conditions

$$G(T(x), T(y), T(y)) \leq a\{G(x, T(y), T(y)) + G(y, T(x), T(x))\} \tag{2.10}$$

or

$$G(T(x), T(y), T(y)) \leq a\{G(x, x, T(y)) + G(y, y, T(x))\}, \tag{2.11}$$

for all  $x, y \in X$ , where  $a \in [0, 1/2)$ , then  $T$  has a unique fixed point (say  $u$ ), and  $T$  is  $G$ -continuous at  $u$ .

*Proof.* Suppose that  $T$  satisfies condition (2.10), then we have

$$\begin{aligned} G(Tx, Ty, Ty) &\leq a\{G(y, Tx, Tx) + G(x, Ty, Ty)\}, \\ G(Ty, Tx, Tx) &\leq a\{G(x, Ty, Ty) + G(y, Tx, Tx)\}, \end{aligned} \tag{2.12}$$

for all  $x, y \in X$ .

Suppose that  $(X, G)$  is symmetric, then by definition of the metric  $(X, d_G)$  and (1.2), we get

$$d_G(Tx, Ty) \leq a\{d_G(x, Ty) + d_G(y, Tx)\} \quad \forall x, y \in X. \tag{2.13}$$

Since  $0 \leq 2a < 1$ , then the existence and uniqueness of the fixed point follow from a theorem in metric space  $(X, d_G)$  (see [13]).

However, if  $(X, G)$  is not symmetric, then by definition of the metric  $(X, d_G)$  and (1.3), we have

$$4a \qquad 4a \qquad d_G(Tx, Ty) \leq \frac{4a}{3} d_G(x, Ty) + \frac{4a}{3} d_G(y, Tx) \quad \forall x, y \in X. \tag{2.14}$$

So, the metric space  $(X, d_G)$  gives no information about this map since  $8a/3$  need not be less than 1. But this can be proved by  $G$ -metric.

Let  $x_0 \in X$  be arbitrary point, and define the sequence  $(x_n)$  by  $x_n = T^n(x_0)$ , then by (2.51), we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq a\{G(x_{n-1}, x_{n+1}, x_{n+1}) + G(x_n, x_n, x_n)\} = aG(x_{n-1}, x_{n+1}, x_{n+1}). \tag{2.15}$$

But

$$G(x_{n-1}, x_{n+1}, x_{n+1}) \leq aG(x_{n-1}, x_n, x_n) + aG(x_n, x_{n+1}, x_{n+1}), \tag{2.16}$$

thus we have

$$G(x_n, x_{n+1}, x_{n+1}) \leq \frac{a}{1-a} G(x_{n-1}, x_n, x_n). \tag{2.17}$$

Let  $k = a/(1 - a)$ , hence  $0 \leq k < 1$  then continue in this procedure, we will get that  $G(x_n, x_{n+1}, x_{n+1}) \leq k^n G(x_0, x_1, x_1)$ .

For all  $n, m \in \mathbf{N}; n < m$ , we have by rectangle inequality

$$G(x_n, x_m, x_m) \leq G(x_n, x_{n+1}, x_{n+1}) + G(x_{n+1}, x_{n+2}, x_{n+2}) + G(x_{n+2}, x_{n+3}, x_{n+3}) + \dots + G(x_{m-1}, x_m, x_m) \leq (k^n + k^{n+1} + \dots + k^{m-1})G(x_0, x_1, x_1)$$

$$\leq \frac{k^n}{1-k} G(x_0, x_1, x_1). \tag{2.18}$$

Then,  $\lim G(x_n, x_m, x_m) = 0$ , as  $n, m \rightarrow \infty$ , and so,  $(x_n)$  is G-Cauchy sequence. By completeness of  $(X, G)$ , there exists  $u \in X$  such that  $(x_n)$  is G-converge to  $u$ .

Suppose that  $T(u) \neq u$ , then

$$G(x_n, T(u), T(u)) \leq a\{G(x_{n-1}, T(u), T(u)) + G(u, x_n, x_n)\}. \tag{2.19}$$

Taking the limit as  $n \rightarrow \infty$ , and using the fact that the function  $G$  is continuous, then  $G(u, T(u), T(u)) \leq aG(u, T(u), T(u))$ . This contradiction implies that  $u = T(u)$ .

To prove uniqueness, suppose that  $u \neq v$  such that  $T(v) = v$ , then  $G(u, v, v) \leq a\{G(u, v, v) + G(v, u, u)\}$ , so

$$G(u, v, v) \leq k = \frac{a}{1-a} G(v, u, u) \tag{2.20}$$

again by the same argument, we can verify that  $G(u, v, v) \leq k^2 G(u, v, v)$ , which implies that  $u = v$ .

**Conclusion**

The study of metric fixed point theory plays an important role because the study finds applications in many important areas as diverse as differential equations, operation research, mathematical economics and the like. Different generalizations of the usual notion of a metric space were proposed by several mathematicians (called 2-metric spaces) and Dhage (called D-metric spaces). K.S.Ha et.al. have pointed out that the results cited by Gahler are independent, rather than generalizations, of the corresponding results in metric spaces. Moreover, it was shown that Dhage’s notion of D-metric space is flawed by errors and most of the results established by him and others are invalid. These facts determined Mustafa and Sims to introduce a new concept in the area, called G-metric space.

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