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A Study of Laplace Transform and Applications to Differential Equations

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ABSTRACT

In general, the differential equation was normally solved. The transformation of Laplace promotes resolution. In various areas of science, engineering and technology, the transformation of Laplace is applied. In so many areas, the Laplace transformation applies. The transformation of the position is implemented by converting to the frequency domain to solve the time domain function. This is where Laplace transformation has been applied in linear ordinary differential equations with constant coefficient and several ordinary equations with varying coefficients. The transformation of the slot simplifies solving problems in engineering applications and facilitates resolution of differential equations. This article discusses Laplace Transform and Differential Equations applications.

1. Introduction

Laplace transformation is a mathematical tool used for the resolution and conversion of differential equations. It is generally efficient to solve either ordinary or partial linear differential equations. It reduces to algebraic an ordinary difference equation. The Laplace Transformation method can easily be used to solve regular linear differential equations with constant coefficients and variable coefficients without finding the overall solution and the arbitrary constant. It is used to solve physical difficulties. This involves an integral and normal equation of differences. It can also be used as an simple and fast way to transform the signal network into a frequency domain. It has wide applications, as well as basic sciences and mathematics, in various fields of engineering and technology.

2. Definition

Let F (t) is a well-defined function of t for all t ≥ 0. The Laplace transformation of F (t), denoted by f (p) or L {F (t)}, is defined as

$$\mathrm{L}\ \{\mathrm{F}\ (\mathrm{t})\} = \int_0^{\infty} e^{-pt}\, F(t)dt = f(p)$$

Provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not.

Where p the parameter which may be real or complex number and L is the Laplace transformation operator. The Laplace transformation of $\mathrm{F}\ (\mathrm{t})$ i.e. $\int_0^{\infty} e^{-pt}\, F(t)dt$ exists for $p>a$, if F (t) is continuous and $\lim_{n\to\infty}\{e^{-at}\, F(t)\}$ is finite. It should however, be keep in mind that above condition are sufficient and not necessary.

Motivation. In several fields of science and engineering, the transformation methods are commonly used. In signal processing, circuit analysis and in probability theory applications , for instance, transform methods are applied. The fundamental idea is to turn a function from its original domain into a transformable domain where such operations can be carried out more efficiently, carried out in the transfer domain and the output (from the transform domain into the original domain) can then be reverse-transformed.

For example, the convolution operation of two functions of time t, f(t) and g(t) is defined as:

$$f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)\cdot g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)\cdot g(\tau)d\tau$$

A real figure with µ. In the Laplace or Fourier domain, the confusion in the time domain becomes multiplication. As an application, the pulse response h(t) and signal x(t) are the input of a linear circuit. The output is then y(t) = x(t) to return the linear circuit. If H(s), X(s) and Y(s) are the impulse response, input, and output transformations in Laplace, then Y(s) = H(s)•X(s) are shown in Figure 1. Once we know Y (s), we can use the inversely transformed laplace to obtain the circuit response, y(t), time t function.

Transform techniques provide a bridge between the common method of variables separation and numerical methods for resolving linear partial differential equations. Although transformational methods may be effective in a wider range of problems in a certain way similar to the separation of variables. Though analytically, numerical and asymptotic techniques can not be found even when the opposite of the transform.

3. Laplace Transform.

Let R be the field of real numbers and C the field of complex numbers. Consider a function f : R 7→ R such that f(t), t ∈ R, t ≥ 0. Then the Laplace Transform of f(t) is denoted as L[f(t)] and it is defined as F(s) with s ∈ C:

$$\boxed{F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt.}$$

The Laplace transform F(s) typically exists for all complex numbers s such that Re(s) > a where a ∈ R is a constant which depends on the behavior of f(t).

The Inverse Laplace Transform is given by the following complex integral:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) ds$$

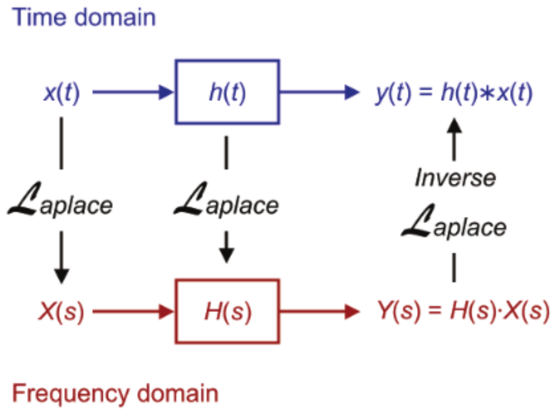

Figure 1: A circuit with the impulse response h(t) and x(t) as input. Then the output is y(t) = x(t) ∗ h(t). If H(s), X(s) and Y (s) are respectively the Laplace transforms of the impulse response, the input, and the output, then Y (s) = H(s) · X(s)

where γ is a real number so that the contour path of integration is in the region of convergence of F(s) normally requiring sρ> Re(sρ) for every singularity sρ of F(s) and with i = √ −1. If all singularities are in the left half-plane (in this case Re(sρ) < 0 for every sρ), then γ can be set to zero and the above inverse integral formula becomes identical to the inverse Fourier transform. This integral is known as the Fourier-Mellin integral.

The Bilateral Laplace Transform is defined as:

$$F(s) = \mathcal{L}[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(t)dt.$$

In probability theory, the Laplace transform is defined by means of an expectation value. If *X* is a random variable with probability density function f_X, then the Laplace transform of f_X is given by the expectation:

$$(\mathcal{L}f_X)(s) = E[e^{-sX}].$$

Instead, it was often referred to as a random variable X transformation of Laplace through abuse of language. The variable s replaced with −t provides an X function for generating the moment. The transformation of Laplace applies to the entire probability theory, including the first transition of stochastic processes such as Markov 's chains and renewal theory.

Laplace transformation of elementary function:

1) $L\{1\} = \frac{1}{p}, \quad p > 0$
2) $L\{t^n\} = \frac{n!}{p^{n+1}}, \quad \text{where } n = 0,1,2,3 \ldots \ldots \ldots.$
3) $L\{e^{at}\} = \frac{1}{p-a}, \quad p > a$
4) $L\{\text{sinat}\} = \frac{a}{p^2+a^2}, \quad p > 0$
5) $L\{\text{sinhat}\} = \frac{a}{p^2-a^2}, \quad p > |a|$
6) $L\{\text{cosat}\} = \frac{p}{p^2+a^2}, \quad p > 0$
7) $L\{\text{coshat}\} = \frac{p}{p^2-a^2}, \quad p > |a|$

Proof: By the definition of Laplace transformation, we know that

1) L $\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$ then

L $\{1\} = \int_0^\infty e^{-pt}\, 1\, dt$

$= -\frac{1}{p}\,(e^{-\infty} - e^{-0}) = \frac{1}{p}(0-1)$

$= \frac{1}{p} = f(p)\ , p > 1$

2) L $\{F(t)\} = \int_0^\infty e^{-pt} F(t)dt$ then.

L $\{\,sinhat\,\} = \int_0^\infty e^{-pt}\, sinhat\; dt$

$$\int_0^\infty e^{-pt}\left(\frac{e^{at} - e^{-at}}{2}\right)dt$$

$$\int_0^\infty \left(\frac{e^{-(p-a)t} - e^{-(p+a)t}}{2}\right)dt$$

$= -\frac{1}{2(p-a)}\,(e^{-\infty} - e^{-0})$

$+\frac{1}{2(p+a)}\,(e^{-\infty} - e^{-0})$

$= \frac{1}{2(p-a)} - \frac{1}{2(p+a)}$

$= \frac{1}{2} \cdot \frac{2a}{p^2-a^2}$

Therefore, $L\,\{sinhat\} = \frac{a}{p^2-a^2}\ ,\ \ p > |a|$

Solution of Ordinary Differential Equations with Variable Coefficients

Laplace Transformation of derivatives: Let F is an exponential order, and that F is a continuous and F is piecewise continuous on any interval, than $L\,\{F'(t)\} = \int_0^\infty e^{-pt}\, F'(t)dt$

$= [0 - F(0)] - \int_0^\infty -pe^{-pt}\, F(t)dt \quad = -F(0) + p\int_0^\infty e^{-pt}\, F(t)dt$

$= \quad pL\{F(t)\} - F(0)$

$= \quad pf(p) - F(0)$

Now, since $L\,\{F'(t)\} = pL\{F(t)\} - F(0)$

Therefore, $L\,\{F''(t)\} = pL\{F'(t)\} - F'(0)$

$L\,\{F''(t)\} = p\,\{pL\{F(t)\} - F(0)\} - F'(0)$

$L\,\{F''(t)\} = p^2L\{F(t)\} - F(0) - F'(0)$

$L\,\{F''(t)\} = p^2f(p) - F(0) - F'(0)$

Similarly $L\,\{F'''(t)\} = p^3f(p) - p^2F(0) - pF'(0) - F''(0)$

and so on……………………..

It is an important part for the solution of differential equation s, it is very helpful.

Now the Laplace transformation of derivatives, linearity property, T propagation and reverse layout transformation will be the solution to resolve the difference equation with the Variable Coefficient.

1) *Example-1: Solve*

$$t\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + tx = cost\ with\ condition\ x(0) = 1$$

Solution: Given equation is

$$t\,x'' + 2x' + tx = cost$$

Taking Laplace Transformation on both sides

$$L\{\,tx''\}+2L\{x'\}+L\{tx\}=L\{cost\}$$

Using $If\ \ L\{f(t)\}=f(p),\ then\ =\ \ L\{t^n[f(t)]\}=\left[\frac{d^n}{ds^n}\{f(p)\}\right]$

Now

$$p^2x(p)-px(0)-x'(0)+9x(p)=\frac{p}{p^2+4}$$

Using condition $x(0)=1,\ x\left(\frac{\pi}{2}\right)=-1$

$$x(p)(p^2+9)-p-A=\frac{p}{p^2+4}$$

$$x(p)=\frac{p+A}{p^2+9}+\frac{p}{(p^2+4)(p^2+9)}$$

$$x(p)=\frac{p+A}{p^2+9}+\frac{p}{(p^2+4)(p^2+9)}$$

By partial fraction

$$x(p)=\frac{p}{p^2+9}+\frac{A}{p^2+9}+\frac{p}{5(p^2+4)}-\frac{p}{5(p^2+9)}$$

$$L\{x\}=\frac{p}{p^2+9}+\frac{A}{p^2+9}+\frac{p}{5(p^2+4)}-\frac{p}{5(p^2+9)}$$

$$x=L^{-1}\left\{\frac{p}{p^2+9}\right\}+L^{-1}\left\{\frac{A}{p^2+9}\right\}+L^{-1}\left\{\frac{p}{5(p^2+4)}\right\}-L^{-1}\left\{\frac{p}{5(p^2+9)}\right\}$$

$$x=cos3t+\frac{A}{3}sin3t+\frac{1}{5}cos2t-\frac{1}{5}cos3t$$

$$x=\frac{A}{3}sin3t+\frac{1}{5}cos2t+\frac{4}{5}cos3t$$

Since $x\left(\frac{\pi}{2}\right)=-1$

$$-1=-\frac{A}{3}-\frac{1}{5}$$

$$A=\frac{12}{5}$$

Hence the required solution is

$$x=\frac{4}{5}sin3t+\frac{1}{5}cos2t+\frac{4}{5}cos3t$$

2) *Example-2: Solve*

$$t\,x''+(1-2t)x'-2x=0,\qquad if$$

Solution: $t\,x''+(1-2t)x'-2x=0$

Taking Laplace Transformation on both sides

$$L\{\,tx''\}+L[(1-2t)\{x'\}]-2L\{x\}=0$$

$$L\{\,tx''\}+L\{x'\}-2L\{tx'\}-2L\{x\}=0$$

$$-\frac{d}{dp}L\{\,x''\}+L\{x'\}+2\frac{d}{ds}L\{x'\}-2L\{x\}=0$$

$$-\frac{d}{dp}[p^2x(p)-px(0)-x'(0)]+[px(p)-x(0)]+2\frac{d}{ds}[px(p)-x(0)]-2x(p)=0$$

$$-\frac{d}{dp}[p^2x(p)-p-2]+[px(p)-1]+2\frac{d}{dp}[px(p)-1]-2x(p)=0$$

$$-\left[p^2\frac{d\,x(p)}{dp}+2px(p)-1\right]+[px(p)-1]+2\left[p\frac{d\,x(p)}{dp}+x(p)\right]-2x(p)=0$$

$$-(p-2)\frac{d\,x(p)}{dp} = x(p)$$

$$\frac{d\,x(p)}{dp} + \frac{1}{p-2}dp = 0$$

Integration on both sides with respect to p

$$\log x(p) + \log(p-2) = logk$$

$$x(p) = \frac{k}{p-2}$$

Taking inverse Laplace transformations on both sides

$$L^{-1}\{x(p)\} = L^{-1}\{\frac{k}{p-2}\}$$

$$x(t) = ke^{2t}$$

But $x(0) = 1$ *putting in above*

then $k = 1$

$$x(t) = e^{2t}$$

This is the required solution.

Laplace Transforms of a few functions f(t).

In each case we start from the definition. For example, f(t) = k:

$$F(s) = \mathcal{L}\left[e^{\alpha t}\right] = \int_0^\infty e^{-st}e^{\alpha t}dt = -\frac{1}{s-\alpha}\left[e^{-(s-\alpha)t}\right]_{t=0}^{t\mapsto\infty} = \frac{1}{s-\alpha}.$$

Properties of the Laplace Transform.

The properties of the Laplace Transform summarized in Table 1 can be derived easily starting from the definition.

Table 1: Properties of the Laplace Transform. The function f(t) is assumed to be n-times differentiable, with n-th derivative of exponential type. Notations: $F(s) = \mathcal{L}[f(t)],\ G(s) = \mathcal{L}[g(t)],\ f^{(n)}$ is the n-th derivative of $f(t),\ F^{(n)}(s)$ is the n-th derivative of F(s), u(t) is the Heaviside step function, $u(t) = \int_{-\infty}^{t}\delta(\tau)d\tau$ with δ the Dirac delta function, (f ∗g)(t) is the convolution of $f(t)$ and $g(t),\ \alpha,\beta\in\mathbb{R}$.

Property	Time-domain	s-domain
Linearity	$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$
Scaling	$f(\alpha t)$	$\frac{1}{\alpha}F(\frac{s}{\alpha})$
Frequency shifting	$e^{\alpha t}f(t)$	$F(s-\alpha)$
Time shifting	$f(t-\alpha)u(t-\alpha)$	$e^{-\alpha s}F(s)$
Frequency differentiation	$t^n f(t)$	$(-1)^n F^{(n)}(s)$
Frequency integration	$f(t)/t$	$\int_s^\infty F(r)dr$
Differentiation	$f^{(n)}(t)$	$s^n F(s) - s^{n-1}f(0) - ... - f^{(n-1)(0)})$
Integration	$\int_0^t f(\tau)d\tau$	$(u * f)(t)\frac{1}{s}F(s)$
Convolution	$(f * g)(t)$	$F(s) * G(s)$
Periodic function $f(t) = f(t+T)$	$f(t)$	$\frac{1}{1-e^{-Ts}}\int_0^T e^{-st}f(t)dt$

For example, to prove linearity consider two functions *f(t)* and *g(t)* and their Laplace Transforms:

$$F(s) = \mathcal{L}\left[f(t)\right] = \int_0^\infty e^{-st}f(t)dt \quad \text{and} \quad G(s) = \mathcal{L}\left[g(t)\right] = \int_0^\infty e^{-st}g(t)dt.$$

From the definition of the Laplace Transform it follows that

$$\mathcal{L}\left[f(t)+g(t)\right] = \int_0^\infty e^{-st}\left[f(t)+g(t)\right]dt = \int_0^\infty e^{-st}f(t)dt + \int_0^\infty e^{-st}g(t)dt = F(s)+G(s).$$

It is also easy to see that *F*(*0*) represents the area under the curve *f*(*t*):

$$F(s=0)\int_0^\infty f(t)dt$$

The Laplace Transform can be expressed as:

$$\boxed{\mathcal{L}\left[f(t)\right] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \dots}$$

Proof: This important property of the Laplace Transform is a consequence of the following equality:

$$\int e^{-\alpha x} f(x)dx = -\frac{e^{-\alpha x}}{\alpha}\left[f(x) + \frac{f'(x)}{\alpha} + \frac{f''(x)}{\alpha^2} + \frac{f'''(x)}{\alpha^3} + \dots\right]$$

This is easy to prove by applying the derivation operator of both sides; then the left hand side becomes $A = e^{-\alpha x} f(x)$. The right hand side is the sum of two terms *B* and *C*:

$$B = \alpha\frac{e^{-\alpha x}}{\alpha}\left[f(x) + \frac{f'(x)}{\alpha} + \frac{f''(x)}{\alpha^2} + \frac{f'''(x)}{\alpha^3} + \dots\right]$$

$$C = -\frac{e^{-\alpha x}}{\alpha}\left[f'(x) + \frac{f''(x)}{\alpha} + \frac{f'''(x)}{\alpha^2} + \frac{f^{iv}(x)}{\alpha^3} + \dots\right]$$

Then

$$B + C = e^{-\alpha x} f(x).$$

Thus A = B + C and this equality allows us to express the Laplace Transform as:

$$\mathcal{L}\left[f(t)\right] = \left\{-\frac{e^{-st}}{s}\left[f(t) + \frac{f'(t)}{s} + \frac{f''(t)}{s^2} + \frac{f'''(t)}{s^3} + \dots\right]\right\}_0^\infty.$$

But,

$$\lim_{t\to\infty}\left\{-\frac{e^{-st}}{s}\left[f(t) + \frac{f'(t)}{s} + \frac{f''(t)}{s^2} + \frac{f'''(t)}{s^3} + \dots\right]\right\} = 0$$

After we subtract the value of the expression for t = 0 we obtain the result enounced:

$$\mathcal{L}\left[f(t)\right] = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \frac{f''(0)}{s^3} + \frac{f'''(0)}{s^4} + \dots$$

4. Conclusion

Here we have used Laplace transformations to solve the differential equations in different areas of differential equations which have a variable coefficient with limits. Laplace 's primary use is to transform the time domain functions into frequency domain features. Laplace basic function transformation and Laplace transformation derivatives have been studied in depth and are highly effective instruments to simplify differential equations.

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