RESEARCH REVIEW JOURNALS
Volume-04
Issue-03
March -2019

ISSN: 2455-3085 (Online)
RESEARCH REVIEW International Journal of Multidisciplinary
www.rrjournals.com [UGC Listed Journal]

Fractional Derivative and Integral Formulas Involving By Generalized Hypergeometric Function

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ARTICLE DETAILS

Article History
Published Online: 13 March 2019

Keywords
Pochhemmer symbol, Fractional Calculus, Fractional Derivative, Fractional Integration, Beta Transform, Laplace Transform, Whittaker Transform

MSC 2010: 26A33, 33C45, 33C60, 33C70

ABSTRACT

In this paper, some new formulae are established by employing the Riemann-Liouville fractional derivative and integral operators involving new extended Gauss hypegeometric function and Confluent hypergeometric function. Further using the integral transforms of the new finding of this paper, new image formulae are developed. The results obtained here are quite general in nature and capable of yielding a very large number of known and presumably new results.

1. Introduction and preliminaries

Recently, Mohd et al. [1] introduced the extension of Beta Function in terms of Mittag-Leffler function, which is defined as:

$$B_{\alpha}^{p}(x,y) := \int_{0}^{1} t^{x-1}(1-t)^{y-1} E_{\alpha}\left(-\frac{p}{t(1-t)}\right) dt \quad \left(\alpha \in R_{0}^{+}, R(p) \geq 0\right) \tag{1.1}$$

Later on, they used the above definition of extended Beta function with extended Gauss hypergeometric function and the extended confluent hypergeometric function and obtained result as follows [1]:

$$F_{\mathrm{p},\alpha}(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}; \left(\alpha \in \square^{+}, \mathrm{p} \in \square_{0}^{+}, |z|<1, \Re(c) > \Re(b) > 0\right) \tag{1.2}$$

$$\Phi_{\mathrm{p},\alpha}(b;c;z) = \sum_{n=0}^{\infty} \frac{B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)} \frac{z^n}{n!}; \left(\alpha \in \square^{+}, \mathrm{p} \in \square_{0}^{+}, \Re(c) > \Re(b) > 0\right). \tag{1.3}$$

The concept of the Hadamard product (or the convolution) of two analytic functions is very useful in this present study. It is helpful to decompose a newly emerging function into two known functions. The Hadamard product is taken as follows

$$f(z) := \sum_{n=0}^{\infty} a_n z^n, \qquad (|z|<R_f) \tag{1.4}$$

and

$$g(z) := \sum_{n=0}^{\infty} b_n z^n, \qquad (|z|<R_g) \tag{1.5}$$

where f(z) and g(z) are two power series whose radii of convergence are denoted by R_f and R_g respectively. Kiryakova, V. and Pohlen, T. [2, 3] worked on this concept. Their convolution is given by

$$(f*g)(z) := \sum_{n=0}^{\infty} a_n b_n z^n = (g*f)(z), \qquad (|z|<R), \tag{1.6}$$

where

$$R = \lim_{n\to\infty}\left|\frac{a_n b_n}{a_{n+1}b_{n+1}}\right| = \lim_{n\to\infty}\left(\left|\frac{a_n}{a_{n+1}}\right|\right).\lim_{n\to\infty}\left(\left|\frac{b_n}{b_{n+1}}\right|\right) = R_f.R_g, \tag{1.7}$$

therefore, in general, $R \geq R_f . R_g$. For various investigations involving the Hadamard product (or the convolution), the interested reader may refer to recent papers on the subject (see, for example, [4, 5] and the references cited there in). Also the Fox Wright function

${}_p\Psi_q$ (z) (p; q $\in \mathbb{N}_0$) with p numerator and q denominator parameters are defined for $a_1,...,a_p \in b_1,...,b_q \in \mathbb{N}_0$ by Kilbas, A.A. et al., Samko , S.G et al. , Srivastava, H. M. and Karlsson, P. W and Mathai , A.M. et al. [6, 7, 8, 9] as

$$ {}_p\Psi_q\left[\begin{matrix}(a_1,\alpha_1),...,(a_p,\alpha_p)\\(b_1,\beta_1),...,(b_q,\beta_q)\end{matrix}\Big|z\right]=\sum_{n=0}^{\infty}\frac{\Gamma(a_1+\alpha_1 n),...,\Gamma(a_p+\alpha_p n)}{\Gamma(b_1+\beta_1 n),...,\Gamma(b_q+\beta_q n)}\frac{z^n}{n!} \quad (1.8)$$

where the coefficients $\alpha_1,...,\alpha_p,\beta_1,...,\beta_q \in \mathbb{R}^+$ are such that

$$1+\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{q}\alpha_j \geq 0 \quad (1.9)$$

The generalized hypergeometric function ${}_pF_q$ (p, q $\in \mathbb{N}_0$) given by Srivastava, H. M. and Karlsson, P. W [8] is as follows

$$ {}_pF_q\left[\begin{matrix}a_1,...,a_p;\\b_1,...,b_q;\end{matrix}\Big|z\right]=\frac{\Gamma(b_1)...\Gamma(b_p)}{\Gamma(a_1)...\Gamma(a_q)}\,{}_p\Psi_q\left[\begin{matrix}(a_1,1),...,(a_p,1);\\(b_1,1),...,(b_q,1);\end{matrix}\Big|z\right] \quad (1.10)$$

1.2 Fractional derivative and integral operators

Right sided Riemann-Liouville fractional integral operator I_{a+}^{σ} , left sided Riemann-Liouville fractional integral operator I_{b-}^{σ} and their corresponding Riemann-Liouville fractional derivative operator D_{a+}^{σ} and D_{b-}^{σ} are given by Samko, S. G. et al. [7] as follows

Lemma 1 If $\Psi=[a,b](-\infty<a<b<\infty)$ be finite interval on the real axis R. The left-sided Riemann-Liouville fractional integral operators I_{a+}^{σ} and right-sided Riemann-Liouville fractional integral operators I_{b-}^{σ} of order $\sigma \in \mathbb{C}$ are defined as:

$$\left(I_{\alpha+}^{\sigma}f\right)(x)=\frac{1}{\Gamma(\sigma)}\int_a^x\frac{f(t)}{(x-t)^{1-\sigma}}dt, \quad (x>a;\Re(\sigma)>0), \quad (1.2.1)$$

$$\left(I_{b-}^{\sigma}f\right)(x)=\frac{1}{\Gamma(\sigma)}\int_x^b\frac{f(t)}{(t-x)^{1-\sigma}}dt, \quad (x<b;\Re(\sigma)>0). \quad (1.2.2)$$

Lemma 2 If $\Psi=[a,b](-\infty<a<b<\infty)$ be finite interval on the real axis R. The left-sided Riemann-Liouville fractional derivative operators D_{a+}^{σ} and right-sided Riemann-Liouville fractional derivative operators D_{b-}^{σ} of order $\sigma \in \mathbb{C}$ are defined as:

$$\left(D_{a+}^{\sigma}f\right)(x)=\frac{d^n}{dx^n}\left(I_{a+}^{n-\sigma}f\right)(x) \quad (\Re(\sigma)\geq 0;n=1+[\Re(\sigma)]), \quad (1.2.3)$$

$$\left(D_{b-}^{\sigma}f\right)(x)=(-1)^n\frac{d^n}{dx^n}\left(I_{b-}^{n-\sigma}f\right)(x) \quad (\Re(\sigma)\geq 0;n=1+[\Re(\sigma)]), \quad (1.2.4)$$

where the function is locally integrable, $\Re(\sigma)$ denotes real part of the complex number and [$\Re(\sigma)$] means greatest integer in $\Re(\sigma)$. Also the following nth order derivative of x^{α} is defined as:

$$\frac{d^n}{dx^n}\left(x^{\alpha}\right)=\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-n)}x^{\alpha-n} \quad \Re(\alpha)>0 \quad (1.2.5)$$

Following result is also required for this present work and is given by Srivastava, H. M., & Choi, J.[10]

$$\int_b^a(a-t)^{\beta-1}(t-b)^{\alpha-1}dt=(a-b)^{\alpha+\beta-1}B(\alpha,\beta), \quad (\Re(\alpha)>0;\Re(\beta)>0;b<a) \quad (1.2.6)$$

2. Fractional integral and derivative formulae involving generalized Hypergeomertric function.

In this section , some new formulae by using the Riemann-Liouville fractional integral and derivative operator involving generalized Hypergeomertric function are derived.

Theorem 2.1. Let, the following integral formula holds for x > a

$$\left(I_{a+}^{\sigma}\left(t-a\right)^{p}F_{p.\alpha}\left(a,b;c;(\xi(t-a)^{\mu}\right)\right)(x)=$$
$$\left(x-a\right)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi\left(x-a\right)^{\mu}\right)*{}_{1}\Psi_{1}\left[\begin{matrix}p+1,\mu\\ \sigma+p+1,\mu\end{matrix};\xi\left(x-a\right)^{\mu}\right] \quad (2.1.1)$$

Proof. Let, us first take the left-hand side of equation (2.1.1). Applying (1.2) and using the equation (1.2.1) and interchanging the order of integration and summation, which is valid under the conditions of theorem 2.1, is given by

$$\frac{1}{\Gamma(\sigma)}\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}\int_{a}^{x}\left(x-t\right)^{\sigma-1}\left(t-a\right)^{p+\mu n}dt \quad (2.1.2)$$

Again applying the result (1.2.6), the above equation (2.1.2) reduces to

$$=\frac{1}{\Gamma(\sigma)}\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}(x-a)^{\sigma+p+\mu n}B\left(\sigma,\mathrm{p}+\mu n+1\right), \quad (2.1.3)$$

After simplification, the above equation (2.1.3) reduces to

$$=\frac{\left(x-a\right)^{\sigma+p}}{\Gamma(\sigma)}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)\Gamma(\sigma)}{\Gamma\left(\sigma+p+\mu n+1\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\}, \quad (2.1.4)$$

The equation (2.1.4) can also be written as

$$=\left(x-a\right)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)}{\Gamma\left(\sigma+p+\mu n+1\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\}, \quad (2.1.5)$$

By applying the Hadamard product from (1.6) in Eq (2.1.5) in the view of (1.2) and (1.10), then the required equation (2.1.1) is obtained.

Theorem 2.2. Let, the following integral formula holds for b> x

$$\left(I_{b-}^{\sigma}\left(b-t\right)^{p}F_{p.\alpha}\left(a,b;c;(\xi(\mathrm{b-t})^{\mu}\right)\right)(x)=$$
$$\left(b-x\right)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi\left(b-x\right)^{\mu}\right)*{}_{1}\Psi_{1}\left[\begin{matrix}p+1,\mu\\ \sigma+p+1,\mu\end{matrix};\xi\left(b-x\right)^{\mu}\right] \quad (2.2.1)$$

Proof. The proof of this theorem is same as that of theorem 2.1, the only difference is taken as b>x.

Theorem 2.3. Let, the following derivative formula holds for x > a

$$\left(D_{a+}^{\sigma}\left(t-a\right)^{p}F_{p.\alpha}\left(a,b;c;(\xi(t-a)^{\mu}\right)\right)(x)=$$
$$\left(x-a\right)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi\left(x-a\right)^{\mu}\right)*{}_{1}\Psi_{1}\left[\begin{matrix}p+1,\mu\\ -\sigma+p+1,\mu\end{matrix};\xi\left(x-a\right)^{\mu}\right] \quad (2.3.1)$$

Proof. Let, us first use left-hand side of equation (2.3.1). Applying (1.2) and using the equation (1.2.3) and interchanging the order of integration and summation, as follows

$$=\frac{1}{\Gamma(n-\sigma)}\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}\frac{d^{n}}{dx^{n}}\int_{a}^{x}\left(x-t\right)^{n-\sigma-1}\left(t-a\right)^{p+\mu n}dt \quad (2.3.2)$$

And applying the result (1.2.6), the above equation (2.3.2) reduces to

$$=\frac{1}{\Gamma(n-\sigma)}\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}B\left(n-\sigma,\mathrm{p}+\mu n+1\right)\frac{d^{n}}{dx^{n}}(x-a)^{n-\sigma+p+\mu n}, \quad (2.3.3)$$

After simplification, the above equation (2.3.3) reduces to

$$=\frac{(x-a)^{\sigma+p}}{\Gamma(n-\sigma)}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)\Gamma(n-\sigma)}{\Gamma(n-\sigma+p+\mu n+1)}\frac{\Gamma(p-\sigma+n+\mu n+1)}{\Gamma(p-\sigma+n+\mu n+1-n)}\frac{\left(\xi(x-a)^\mu\right)^n}{n!}\right\},$$

(2.3.4)

The above equation (2.3.4) can be written as

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(-\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^\mu\right)^n}{n!}\right\},$$

(2.3.5)

By applying the Hadamard product from (1.6) in Eq (2.3.5) in the view of (1.2) and (1.10), then required result is obtained.

Theorem 2.4. Let, the following derivative formula holds for b> x

$$\left(D_{b-}^{\sigma}(t-a)^p F_{p.\alpha}\left(a,b;c;(\xi(b-t)^\mu\right)\right)(x)=$$

$$(b-x)^{\sigma+p} F_{p,\alpha}\left(a,b;c;\xi(b-x)^\mu\right)*{}_1\Psi_1\left[\begin{matrix}p+1,\mu\\-\sigma+p+1,\mu\end{matrix};\xi(b-x)^\mu\right]$$

(2.4.1)

Proof. The proof of the theorem 2.4 is similar as that of theorem 2.3, only difference is that new b>x is taken.

Theorem 2.5. Let, the following integral formula holds for a<x

$$\left(I_{a+}^{\sigma}(t-a)^p \Phi_{p.\alpha}\left(b;c;(\xi(t-a)^\mu\right)\right)(x)=$$

(2.5.1)

$$(x-a)^{\sigma+p}\Phi_{p,\alpha}\left(b;c;\xi(x-a)^\mu\right)*{}_1\Psi_1\left[\begin{matrix}p+1,\mu\\\sigma+p+1,\mu\end{matrix};\xi(x-a)^\mu\right]$$

Proof. Let, us take left-hand side of equation (2.5.1). Applying (1.3) and using the equation (1.2.1) and interchanging the order of integration and summation, the required equation can be given as

$$\frac{1}{\Gamma(\sigma)}\sum_{n=0}^{\infty}\frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\xi^n}{n!}\int_a^x(x-t)^{\sigma-1}(t-a)^{p+\mu n}\,dt$$

(2.5.2)

Again applying the result (1.2.6), the above equation (2.5.2) reduces to

$$=\frac{1}{\Gamma(\sigma)}\sum_{n=0}^{\infty}\frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\xi^n}{n!}(x-a)^{\sigma+p+\mu n}B(\sigma,\mathrm{p}+\mu n+1),$$

(2.5.3)

After simplification, the above equation (2.5.3) reduces to

$$=\frac{(x-a)^{\sigma+p}}{\Gamma(\sigma)}\left\{\sum_{n=0}^{\infty}\frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)\Gamma(\sigma)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^\mu\right)^n}{n!}\right\},$$

(2.5.4)

The above equation (2.5.4) can be written as $=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^\mu\right)^n}{n!}\right\},$

(2.5.5)

By applying the Hadamard product form (1.6) in the Eq (2.5.5) in the view of (1.3) and (1.10), then required result is obtained.

Theorem 2.6. Let, following integral formula holds for b > x

$$\left(I_{b-}^{\sigma}(b-t)^\rho \Phi_{p.\alpha}\left(b;c;(\xi(b-t)^\mu\right)\right)(x)=$$

$$(b-x)^{\sigma+p}\Phi_{p,\alpha}\left(b;c;\xi(b-x)^\mu\right)*{}_1\Psi_1\left[\begin{matrix}p+1,\mu\\\sigma+p+1,\mu\end{matrix};\xi(b-x)^\mu\right]$$

(2.6.1)

Proof. The proof of the theorem 2.6 is similar as that of theorem 2.5, only difference is of left-sided and right sided integral.

Theorem 2.7. Let, the following derivative formula holds for a>x

$$\left(D_{a+}^{\sigma}\left(t-a\right)^{p}\Phi_{p.\alpha}\left(b;c;(\xi(t-a)^{\mu})\right)\right)(x)=$$

$$\left(x-a\right)^{\sigma+p}\Phi_{p,\alpha}\left(b;c;\xi\left(x-a\right)^{\mu}\right)*{}_1\Psi_1\left[\begin{matrix}p+1,\mu\\-\sigma+p+1,\mu\end{matrix}\;;\xi\left(x-a\right)^{\mu}\right] \quad (2.7.1)$$

Proof. Let, us take the left-hand side of equation (2.7.1) and applying (1.3) and using the equation (1.2.3) and after that interchanging the order of integration and summation, the equation obtained as follows

$$=\frac{1}{\Gamma(n-\sigma)}\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}\frac{d^{n}}{dx^{n}}\int_{a}^{x}\left(x-t\right)^{n-\sigma-1}\left(t-a\right)^{p+\mu n}dt \quad (2.7.2)$$

And applying the result (1.2.6), the above equation (2.7.2) reduces to

$$=\frac{1}{\Gamma(n-\sigma)}\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\xi^{n}}{n!}B\left(n-\sigma,\mathrm{p}+\mu n+1\right)\frac{d^{n}}{dx^{n}}(x-a)^{n-\sigma+p+\mu n}, \quad (2.7.3)$$

After simplification, the above equation (2.7.3) reduces to

$$=\frac{\left(x-a\right)^{\sigma+p}}{\Gamma(n-\sigma)}\left\{\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)\Gamma\left(n-\sigma\right)}{\Gamma\left(-\sigma+p+\mu n+1\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\}, \quad (2.7.4)$$

The above equation (2.7.4) can be written as

$$=\left(x-a\right)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)}{\Gamma\left(-\sigma+p+\mu n+1\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\}, \quad (2.7.5)$$

By applying the Hadamard product from (1.6) in Eq (2.7.5) in the view of (1.3) and (1.10), the required equation (2.7.1) is obtained.

Theorem 2.8. Let, the following derivative formula holds for b> x

$$\left(D_{b-}^{\sigma}\left(t-a\right)^{p}\Phi_{\mathrm{p}.\alpha}\left(b;c;(\xi(\mathrm{b}-t)^{\mu})\right)\right)(x)=$$

$$\left(b-x\right)^{\sigma+p}\Phi_{\mathrm{p},\alpha}\left(b;c;\xi\left(b-x\right)^{\mu}\right)*{}_1\Psi_1\left[\begin{matrix}p+1,\mu\\-\sigma+p+1,\mu\end{matrix}\;;\xi\left(b-x\right)^{\mu}\right] \quad (2.8.1)$$

Proof. The proof of the theorem 2.8 is similar as that of theorem 2.7.

3. Image Formulas Associated With Integral Transform

In this section, certain theorems involving the results obtained in previous section associated with the integral transforms like, Beta transform, Laplace transform and Whittaker transform are established.

3.1. Beta Transform

The Beta transform of f(z) is defined by[11]as

$$B\left\{f(z):\alpha,\beta\right\}=\int_{0}^{1}z^{\alpha-1}(1-z)^{\beta-1}f(z)dz, \quad (3.1)$$

Theorem 3.1.1. Let, following integral formula holds for x > a

$$B\left\{\left(I_{a+}^{\sigma}\left(t-a\right)^{\rho}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$\left(x-a\right)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi\left(x-a\right)^{\mu}\right)*{}_1\Psi_1\left[\begin{matrix}\left(p+1,\mu\right),\left(\alpha,\mu r\right)\\\left(\sigma+p+1,\mu\right),\left(\alpha+\beta,\mu r\right)\end{matrix}\;;\xi\left(x-a\right)^{\mu}\right] \quad (3.1.1.1)$$

Proof. For convenience, first take the left-hand side of the result (3.1.1.1), then using the definition of beta transform in (3.1), obtained as

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\left(I_{a+}^{\sigma}(t-a)^p F_{p,\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x)dz,$$

(3.1.1.2)

Further, using the result from equation (2.1.5) into the above equation (3.1.1.2), then interchanging the order of integration and summation, as follows

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^n}{n!}\right\}\int_0^1 z^{\alpha+\mu n-1}(1-z)^{\beta-1}dz$$

(3.1.1.3)

And applying the definition of beta transform and after simplification, the result obtained as

$$=(x-a)^{\sigma+\rho}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_{\alpha}^{\rho}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\Gamma(\alpha+\mu n)}{\Gamma(\alpha+\beta+\mu n)}\frac{\left(\xi(x-a)^{\mu}\right)^n}{n!}\right\},$$

(3.1.1.4)

By applying the Hadamard product from (1.6) in Equation (3.1.1.4) in the view of (1.2) and (1.10), then the required result is obtained.

Theorem 3.1.2. Let, following integral formula holds for b > x

$$B\left\{\left(I_{b-}^{\sigma}(t-a)^p F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$(b-x)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi(b-x)^{\mu}\right)*{}_1\Psi_1\left[\begin{matrix}(p+1,\mu),(\alpha,\mu)\\(\sigma+p+1,\mu),(\alpha+\beta,\mu)\end{matrix};\xi(b-x)^{\mu}\right]$$ (3.1.2.1)

Proof. The proof of the theorem 3.1.2 is similar as that of theorem 3.1.1

Theorem 3.1.3 Let, following derivative formula holds for x > a

$$B\left\{\left(D_{a+}^{\sigma}(t-a)^p F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

(3.1.3.1)

$$(x-a)^{\sigma+p}F_{p,\alpha}\left(a,b;c;\xi(x-a)^{\mu}\right)*{}_1\Psi_1\left[\begin{matrix}(p+1,\mu),(\alpha,\mu)\\(-\sigma+p+1,\mu),(\alpha+\beta,\mu)\end{matrix};\xi(x-a)^{\mu}\right]$$

Proof. For convenience, first using the left-hand side of the result (3.1.3.1), then using the definition of beta transform (3.1), as follows

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\left(D_{a+}^{\sigma}(t-a)^p F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x)dz,$$

(3.1.3.2)

Further, using the result from equation (2.3.5) into the above equation (3.1.3.2), then interchanging the order of integration and summation, the obtained result is as follows

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(-\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^n}{n!}\right\}\int_0^1 z^{\alpha+\mu n-1}(1-z)^{\beta-1}dz$$

(3.1.3.3)

Again applying the definition of beta transform and after simplification, the above equation (3.1.3.3) will reduces to

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(-\sigma+p+\mu n+1)}\frac{\Gamma(\alpha+\mu n)}{\Gamma(\alpha+\beta+\mu n)}\frac{\left(\xi(x-a)^{\mu}\right)^n}{n!}\right\},$$

(3.1.3.4)

By applying the Hadamard product from (1.6) in Eq (3.1.3.4) in the view of (1.2) and (1.10), the required equation (3.1.3.1) is obtained.

Theorem 3.1.4. Let, following derivative formula holds for b > x

$$B\left\{\left(D_{b-}^{\sigma}(t-a)^{p} F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$(b-x)^{\sigma+p} F_{p,\alpha}\left(a,b;c;\xi(b-x)^{\mu}\right) * {}_{1}\Psi_{1}\left[\begin{matrix}(p+1,\mu),(\alpha,\mu) \\ (-\sigma+p+1,\mu),(\alpha+\beta,\mu)\end{matrix} ;\xi(b-x)^{\mu}\right] \quad (3.1.4.1)$$

Proof. The proof of the theorem 3.1.4 is similar as that of theorem 3.1.3.

3.2 Laplace Transform

The Laplace transform of f(z) is defined by [11] as

$$L\{f(z)\}=\int_{0}^{\infty} e^{-sz} f(z)dz \quad (3.2)$$

Theorem 3.2.1. Let, following integral formula holds for x > a

$$L\left\{z^{l-1}\left(I_{a+}^{\sigma}(t-a)^{p} F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$\frac{(x-a)^{\sigma+p}}{s^{l}} F_{p,\alpha}\left(a,b;c;\xi(x-a)^{\mu}\right) * {}_{2}\Psi_{1}\left[\begin{matrix}(p+1,\mu),(1,\mu) \\ (\sigma+p+1,\mu),\end{matrix} ;\xi(x-a)^{\mu}\right] \quad (3.2.1.1)$$

Proof. For convenience, at first left-hand side of the result (3.2.1.1) is taken, then using the definition of laplace transform (3.2), the obtained result is as follows

$$\int_{0}^{1} e^{-sz} z^{l-1}\left(I_{a+}^{\sigma}(t-a)^{p} F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right]\right)(x)dz, \quad (3.2.1.2)$$

Further, using the result from equation (2.1.5) into the above equation (3.2.1.2) and then interchanging the order of integration and summation, the obtained result is as follows

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n} B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}\int_{0}^{\infty} e^{-sz} z^{l+\mu n-1}dz \quad (3.2.1.3)$$

Again applying the definition of laplace transform and after simplification, the obtained result is given below

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n} B_{\alpha}^{p}(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\Gamma(l+\mu n)}{(s^{l})}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}, \quad (3.2.1.4)$$

By applying the Hadamard product from (1.6) in Eq (3.2.1.4) in the view of (1.2) and (1.10), the required equation (3.2.1.1) is developed.

Theorem 3.2.2. Let, following integral formula holds for b > x

$$L\left\{z^{l-1}\left(I_{b-}^{\sigma}(t-a)^{p} F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$\frac{(b-x)^{\sigma+p}}{s^{l}} F_{p,\alpha}\left(a,b;c;\xi(b-x)^{\mu}\right) * {}_{2}\Psi_{1}\left[\begin{matrix}(p+1,\mu),(1,\mu) \\ (\sigma+p+1,\mu),\end{matrix} ;\xi(b-x)^{\mu}\right] \quad (3.2.2.1)$$

Proof. The proof of the theorem 3.2.2 is similar as that of theorem 3.2.1.

Theorem 3.2.3. Let, following derivative formula holds for x > a

$$L\left\{z^{l-1}\left(D_{a+}^{\sigma}\left(t-a\right)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$
$$\frac{\left(x-a\right)^{\sigma+p}}{s^{l}}F_{p,\alpha}\left(a,b;c;\xi\left(x-a\right)^{\mu}\right)*{}_{2}\Psi_{1}\left[\begin{matrix}(p+1,\mu),(1,\mu)\\(-\sigma+p+1,\mu),\end{matrix};\xi\left(x-a\right)^{\mu}\right]$$

(3.2.3.1)

Proof. For convenience, let us take the left-hand side of the result (3.2.3.1), then using the definition of laplace transform(3.2), as follows

$$\int_{0}^{1}e^{-sz}z^{l-1}\left(I_{a+}^{\sigma}\left(t-a\right)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x)dz,$$

(3.2.3.2)

Further, using the result from equation (2.3.5) into the above equation (3.2.3.2), then interchanging the orderof integration and summation, as follows

$$=\left(x-a\right)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)}{\Gamma\left(\sigma+p+\mu n+1\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\}\int_{0}^{\infty}e^{-sz}z^{l+\mu kr-1}dz$$

(3.2.3.3)

Again, applying the definition of laplace transform and after simplification, the following result is obtained

$$=\left(x-a\right)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_{n}B_{\alpha}^{p}\left(b+n,c-b\right)}{B\left(b,c-b\right)}\frac{\Gamma\left(p+\mu n+1\right)}{\Gamma\left(-\sigma+p+\mu n+1\right)}\frac{\Gamma\left(l+\mu n\right)}{\left(s^{l}\right)}\frac{\left(\xi\left(x-a\right)^{\mu}\right)^{n}}{n!}\right\},$$

(3.2.3.4)

By applying the Hadamard product from (1.6) in Eq (3.2.3.4) in the view of (1.2) and (1.10), the required result is obtained .

Theorem 3.2.4. Let, following derivative formula holds for b > x

$$L\left\{z^{l-1}\left(D_{b-}^{\sigma}\left(b-t\right)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(b-t)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$
$$\frac{\left(b-x\right)^{\sigma+p}}{s^{l}}F_{p,\alpha}\left(a,b;c;\xi\left(b-x\right)^{\mu}\right)*{}_{2}\Psi_{1}\left[\begin{matrix}(p+1,\mu),(1,\mu)\\(\sigma+p+1,\mu),\end{matrix};\xi\left(b-x\right)^{\mu}\right]$$ (3.2.4.1)

Proof. The proof of the theorem 3.2.4 is similar as that of theorem 3.2.3.

3.3.Whittaker Transform

The whittaker transform of f(z) is defined by [11] as

$$\int_{0}^{\infty}t^{\alpha-1}e^{-t/2}W_{\tau,\omega}dt=\frac{\Gamma\left(1/2+\omega+\alpha\right)\Gamma\left(1/2-\omega+\alpha\right)}{\Gamma\left(1/2-\tau+\alpha\right)}$$

(3.3)

Theorem 3.3.1. Let, following integral formula holds for x > a

$$\int_{0}^{\infty}z^{\varsigma-1}e^{-\eta z/2}W_{\tau,\omega}\left(\eta z\right)\left\{\left(I_{a+}^{\sigma}\left(t-a\right)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a)))^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$
$$\frac{\left(x-a\right)^{\sigma+p}}{s^{l}}F_{p,\alpha}\left(a,b;c;\xi\left(x-a\right)^{\mu}\right)*{}_{3}\Psi_{2}\left[\begin{matrix}(p+1,\mu),(1/2+\omega+\varsigma,\mu),(1/2-\omega+\varsigma,\mu)\\(\sigma+p+1,\mu),(1/2-\tau+\varsigma,\mu)\end{matrix};\xi\left(x-a\right)^{\mu}\right]$$

(3.3.1.1)

Proof. Let us first take the left-hand side of the result (3.3.1.1) as follows

$$\int_0^\infty z^{\varsigma-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left(I_{a+}^{\sigma}(t-a)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right])(x)dz,$$

(3.3.1.2)

Further, using the result from equation (2.1.5) into the above equation (3.3.1.2) and then interchanging the order of integration and summation, the following equation is obtaind

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}\int_0^1 e^{-\eta z/2}z^{\varepsilon+\mu n-1}W_{\tau,\omega}(\eta z)dz$$

(3.3.1.3)

By Substituting $\eta z = t$ the following equation is obtained

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}$$

$$\frac{1}{\eta^{\varsigma+\mu n}}\int_0^\infty e^{-t/2}t^{\varepsilon+\mu n-1}W_{\tau,\omega}(t)dz$$

(3.3.1.4)

And after simplification, the above equation (3.3.1.4) reduces to

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)\Gamma(1/2+\omega+\varsigma+\mu n)\Gamma(1/2-\omega+\varsigma+\mu n)}{\Gamma(\sigma+p+\mu n+1)\Gamma(1/2-\tau+\varsigma+\mu n)}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}$$

(3.3.1.5)

By applying the Hadamard product from (1.6) in Eq (3.3.1.5) in the view of (1.2) and (1.10), the required equation (3.3.1.1) is obtained.

Theorem 3.3.2 Let, following integral formula holds for b > x

$$\int_0^\infty z^{\varsigma-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{b-}^{\sigma}(b-t)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(b-t))\right)^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$\frac{(b-x)^{\sigma+p}}{s^{l}}F_{p,\alpha}\left(a,b;c;\xi(b-x)^{\mu}\right)*{}_3\Psi_2\left[\begin{matrix}(p+1,\mu),(1/2+\omega+\varsigma,\mu),(1/2-\omega+\varsigma,\mu)\\(\sigma+p+1,\mu),(1/2-\tau+\varsigma,\mu)\end{matrix};\xi(b-x)^{\mu}\right] \quad (3.3.2.1)$$

Proof. The proof of the theorem 3.3.2 is similar as that of theorem 3.3.1

Theorem 3.3.3 Let, following derivative formula holds for x>a

$$\int_0^\infty z^{\varsigma-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(D_{a+}^{\sigma}(t-a)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right]\right)(x):\alpha,\beta\right\}=$$

$$\frac{(x-a)^{\sigma+p}}{s^{l}}F_{p,\alpha}\left(a,b;c;\xi(x-a)^{\mu}\right)*{}_3\Psi_2\left[\begin{matrix}(p+1,\mu),(1/2+\omega+\varsigma,\mu),(1/2-\omega+\varsigma,\mu)\\(-\sigma+p+1,\mu),(1/2-\tau+\varsigma,\mu)\end{matrix};\xi(x-a)^{\mu}\right] \quad (3.3.3.1)$$

Proof. Let, us first take the left-hand side of the result (3.3.3.1)

$$\int_0^\infty z^{\varsigma-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left(D_{a+}^{\sigma}(t-a)^{p}F_{p.\alpha}(a,b;c;\left[\xi(z(t-a))\right)^{\mu}\right])(x)dz,$$

(3.3.3.2)

Further, using the result from equation (2.3.5) into the above equation (3.3.3.2) and then interchanging the order of integration and summation, the obtained equation is as follows

$$=(x-a)^{\sigma+p}\left\{\sum_{n=0}^{\infty}\frac{(a)_n B_\alpha^p(b+n,c-b)}{B(b,c-b)}\frac{\Gamma(p+\mu n+1)}{\Gamma(-\sigma+p+\mu n+1)}\frac{\left(\xi(x-a)^{\mu}\right)^{n}}{n!}\right\}\int_0^1 e^{-\eta z/2}z^{\varepsilon+\mu n-1}W_{\tau,\omega}(\eta z)dz$$

(3.3.3.3)

By Substituting $\eta z = t$, the obtained equation is as follows

$$= (x-a)^{\sigma+p} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n B_\alpha^p (b+n, c-b)}{B(b, c-b)} \frac{\Gamma(p+\mu n+1)}{\Gamma(-\sigma+p+\mu n+1)} \frac{\left(\xi (x-a)^\mu\right)^n}{n!} \right\}$$

$$\times \frac{1}{\eta^{\varsigma+\mu n}} \int_0^\infty e^{-t/2} t^{\varepsilon+\mu n-1} W_{\tau,\omega}(t) dz$$

(3.3.3.4)

And after simplification, the above equation (3.3.3.4) reduces to

$$= (x-a)^{\sigma+p} \left\{ \sum_{n=0}^{\infty} \frac{(a)_n B_\alpha^p (b+n, c-b)}{B(b, c-b)} \frac{\Gamma(p+\mu n+1)\Gamma(1/2+\omega+\varsigma+\mu n)\Gamma(1/2-\omega+\varsigma+\mu n)}{\Gamma(-\sigma+p+\mu n+1)\Gamma(1/2-\tau+\varsigma+\mu n)} \frac{\left(\xi (x-a)^\mu\right)^n}{n!} \right\}$$

(3.3.3.5)

By applying the Hadamard product from (1.6) in Eq (3.3.3.5) in the view of (1.3) and (1.10), the required result is obtained.

Theorem 3.3.4. Let, following derivative formula holds for b>x

$$\int_0^\infty z^{\varsigma-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left(D_{b-}^\sigma (b-t)^p F_{p.\alpha}(a,b;c;\left[\xi (z(\mathrm{b}-\mathrm{t})))^\mu\right]\right)(x) : \alpha, \beta \right\} =$$

$$\frac{(b-x)^{\sigma+p}}{s^l} F_{p,\alpha}\left(a,b;c;\xi (b-x)^\mu\right) * {}_3\Psi_2 \left[\begin{matrix} (p+1,\mu),(1/2+\omega+\varsigma,\mu),(1/2-\omega+\varsigma,\mu) \\ (\sigma+p+1,\mu),(1/2-\tau+\varsigma,\mu) \end{matrix} ; \xi (b-x)^\mu \right] \quad (3.3.4.1)$$

Proof. The proof of the theorem 3.3.4 is similar as that of theorem 3.3.3.

Conclusion:

In this present paper, some new results are also obtained by findings some particular parameters like p *and* α are obtained results.

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