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Common Fixed Point Theorem in Generalization of Partial Metric Spaces for Set Valued Mappings

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ABSTRACT
In the setting of symmetric and complete G_p-metric spaces, we prove fixed point and common fixed point results for two pairs of set valued mappings in G_p metric spaces. An example is given to strengthen our results.

1. Introduction and Preliminaries

Fixed point theory is one of the most powerful tools of modern mathematics for solving $Tx = x$ for mapping T defined on subsets of metric spaces and also it is an attractive and interesting subject with a large of number of applications in various fields of mathematics and other branches of sciences like game theory, approximation theory, optimization and variational inequalities. The most famous result in this field was known as the Banach contraction principle (1992) [2] proved a theorem which ensures, under appropriate conditions, the existence and uniqueness of a fixed point. This principle has many generalizations in different ways which established and introduced by several authors.

One such generalizations is a partial metric space which introduced by Matthews[10]. In partial metric spaces, self-distance of an arbitrary point need not to be equal zero.

Definition 1.1.A partial metric on a non empty set X is a function : $X \times X \to \mathbb{R}^+, \mathbb{R}^+ := [0, \infty)$, such that for all $x, y, z \in X$
$P(1)\ x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
$P(2)\ p(x,x) \le p(x,y)$,
$P(3)\ p(x,y) = p(y,x)$,
$P(4)\ p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.
A partial metric space is a pair (X, p) such that X is a non empty set and p is a partial metric on X.

As one of the fruitful generalizations of metric spaces, Gahler[4,5] (called G-metric spaces) and Dhage [5,6] (called D-metric spaces). In 2003, Mustafa and Sims [8] found that most of the claims concerning the fundamental topological properties of D-metric spaces were incorrect. Therefore, they [9] introduced a new structure of generalized metric spaces, which are called G-metric spaces, as a generalization of metric spaces, to develop and introduce a new fixed point theory for various mappings in this new structure.

Definition 1.2. Let X be a non empty set, and let G: $X \times X \times X \to \mathbb{R}^+$ be a function satisfy the following axioms:
$G(1) G(x,y,z) = 0$, iff $x = y = z$,
$G(2) 0 < G(x,y,z)$for all $x, y, z \in X$ with $x \ne y$,
$G(3) G(x,x,y) \le G(x,y,z)$, for all $x, y, z \in X$ with $z \ne y$,
$G(4) G(x,y,z) = G(x,z,y) = G(y,z,x) = \ . \ . \ .$ (symmetry in all three variables),
$G(5) G(x,y,z) \le G(x,a,a) + G(a,y,z)$, for all $x, y, z, a \in X$ (rectangle inequality)
Then the function G is called a generalized metric or more specifically, a $G -$ metric on X, and the pair (X, G) is called a G-metric space.

Recently, Zand and Nezhad[12] introduced a generalization and unification of both partial metric space and G-metric space (generalized metric spaces), by giving the notation of G_p-metric space (generalization of partial metric spaces) in the following way.

Definition 1.3. Let X be a non empty set. Suppose that $G_p : X \times X \times X \to \mathbb{R}^+$ satisfies;
$G_p(1)\ x = y = z$ if $G_p(x,x,x) = G_p(y,y,y) = G_p(z,z,z)$
for all $x, y, z \in X$
$G_p(2)\ 0 \le G_p(x,x,x) \le G_p(x,x,y) \le G_p(x,y,z)$ for all $x, y, z \in X$
$G_p(3) G_p(x,y,z) = G_p(x,z,y) = G_p(y,z,x) = \ . \ . \ .$ (symmetry in all three variables),
$G_p(4) G_p(x,y,z) \le G_p(x,a,a) + G_p(a,y,z) - G_p(a,a,a)$
for all $x, y, z, a \in X$
Then G_p is called a G_p metric on X and (X, G_p) is called a G_p metric space.

Example 1.4. Let $X = [0, \infty)$ and define $G_p(x,y,z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Then (X, G_p) is a G_p metric space but (X, G_p) is not a G-metric space.

Example 1.5. If (X, d) is an ordinary metric space, then (X, d) can define G_p metrics on X by $G_p(x,y,z) = d(x,y) + d(y,z) + d(y,x)$ and $G_p(x,y,z) = \max\{d(x,y), d(y,z), d(z,x)\}$

Proposition 1.6. *Let (X, G_p) is a G_p metric space, then for any $x, y, z \in X$ and $a \in X$, it follows that*

(i) $G_p(x,y,z) \le G_p(x,x,y) + G_p(x,x,z) - G_p(x,x,x)$
(ii) $G_p(x,y,y) \le 2G_p(x,x,y) - G_p(x,x,x)$
(iii) $G_p(x,y,z) \le$
$G_p(x,a,a) + G_p(y,a,a) + G_p(z,a,a) - 2G_p(a,a,a)$
(iv) $G_p(x,y,z) \le G_p(x,a,z) + G_p(a,y,z) - G_p(a,a,a)$

Proposition 1.7. *Every G_p-metric space (X,G_p) defines a metric space (X,D_{G_p}) where*

$$D_{G_p}(x,y) = G_p(x,y,y) + G_p(y,x,x) - G_p(x,x,x - G_p(y,y,y)$$

for all $x,y \in X$

Definition 1.8. Let (X,G_p) be a G_p metric space a sequence $\{x_n\}$ is called a G_p convergent to $x \in X$ if
$\lim_{n,m\to\infty} G_p(x,x_m,x_n) = G_p(x,x,x)$.
A point $x \in X$ is said to be limit point of the sequence $\{x_n\}$ and written $x_n \to x$.
Thus if $x_n \to x$ in a G_p metric space (X,G_p) , then for any $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that $|G_p(x,x_n,x_m) - G_p(x,x,x)| < \varepsilon$, for all $n,m > l$.

Definition 1.9. A G_p metric space (X,G_p) is called a symmetric G_p metric space if $G_p(x,y,y) = G_p(x,x,y)$ for all $x,y \in X$

Proposition 1.10. *Let (X,G_p) is a G_p metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent that*
(i) $\{x_n\}$ *is G_p convergent to x*
(ii) $G_p(x_n,x_n,x) \to G_p(x,x,x)$ *as* $n \to \infty$
(iii) $G_p(x_n,x,x) \to G_p(x,x,x)$ *as* $n \to \infty$

Definition 1.11. Let (X,G_p) be a G_p metric space.
(i) A sequence $\{x_n\}$ is called a G_p Cauchy sequence if and only if $\lim_{n,m\to\infty} G_p(x_n,x_n,x_m)$ exists (and is finite)
(ii) A G_p metric space (X,G_p) is said to be G_p complete if and only if every G_p Cauchy sequence in X is G_p convergent to $x \in X$ such that $G_p(x,x,x) = \lim_{n,m\to\infty} G_p(x_n,x_m,x_m)$.

Lemma 1.12 *Let (X,G_p) be G_p metric space. Then*
(i) If $G_p(x,y,z) = 0 \Rightarrow x = y = z$
(ii) If $x \ne y$, *then* $G_p(x,y,y) > 0$.

Definition 1.13. Let (X,G_p) be a G_p metric space and $:X \to X$ be a given mapping. We say that T is continuous in $x_0 \in X$ if for every sequence x_n in X, we have
(i) x_n converges to x_0 in (X,G_p) implies Tx_n converges to Tx_0 in (X,G_p) .
(ii) x_n converges properly to x_0 in (X,G_p) implies Tx_n converges properly to Tx_0 in (X,G_p) .

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of rigorous research activity. Abbas and Rhoades [3] initiated the study of a common fixed point theory in generalized metric spaces and [11] extend that result to G-metric space.

2. Main Results

In this section we state and prove our main results.

Theorem 2.1. Let (X,G_{p1}) and (Y,G_{p2}) be symmetric and complete G-metric spaces. Let T_1,T_2 and T_3 be mappings of X into $B(Y)$ and S_1,S_2 and S_3 be mappings of Y into $B(X)$ satisfying the inequalities

$$G_{p1}(S_1T_1x,\ S_2T_2x',\ S_3T_3x'') = C\max\begin{Bmatrix} G_{p1}(x,x',x''), G_{p1}(x,S_1T_1x,x''), G_{p1}(x',S_2T_2x',x), \\ Gp_1(x'',S_3T_3x'',x'), G_{p2}(T_1x,T_2x',T_3x'') \end{Bmatrix} \quad (1)$$

$$Gp_2(T_2S_1y,\ T_3S_2y',\ T_1S_3y'') = C\max\begin{Bmatrix} G_{p2}(y,y',y''), G_{p2}(y,T_2S_1y,y''), G_{p2}(y',T_3S_2y',y), \\ G_{p2}(y'',T_1S_3y'',y'), G_{p1}(S_1y,S_2y',S_3y'') \end{Bmatrix} \quad (2)$$

for all $x,x',x'' \in X$ and $y,y',y'' \in Y$ and $0 \le C < 1$. If one of the mapping T_1,T_2,T_3,S_1,S_2 and S_3 is continuous, then S_1T_1, S_2T_2 and S_3T_3 have a unique common fixed point $z \in X$ and T_2S_1, T_3S_2 and T_1S_3 have a common fixed point $w \in$ Y. Further $T_1z = T_2z = T_3z = \{w\}$ and $S_1w = S_2w = S_3w = \{z\}$.

Proof:
Let x be an arbitrary point in X, and define the points
$y_1 \in T_1x,\ x_2 \in S_1y_1,\ y_2 \in T_2x_2,\ x_3 \in S_2y_2,\ y_3 \in T_3x_3,\ x_4 \in S_3y_3.$
Define a sequence $\{x_n\}$ and $\{y_n\}$ in $B(X)$ and $B(Y)$ respectively, by choosing a point,

$$\begin{aligned} y_{3n-2} &\in T_1x_{3n-2} = Y_{3n-2} \\ x_{3n-1} &\in S_1y_{3n-2} = X_{3n-1} \\ y_{3n-1} &\in T_2x_{3n-1} = Y_{3n-1} \\ x_{3n} &\in S_2y_{3n-1} = X_{3n} \\ y_{3n} &\in T_3x_{3n} = Y_{3n} \end{aligned}$$

$x_{3n+1} \in S_3y_{3n} = X_{3n+1}$ for each $n \in \mathbb{N}$

Using (1) and (2), we have

$$\begin{aligned} G_{p1}(x_{3n+1},x_{3n},x_{3n-1}) &= G_{p1}(S_3T_3x_{3n},S_2T_2x_{3n-1},S_1T_1x_{3n-2}) \\ &= G_{p1}(S_1T_1x_{3n-2},S_2T_2x_{3n-1},S_3T_3x_{3n}) \\ &\le C\max\begin{Bmatrix} G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p1}(x,x,x), G_{p1}(x,x,x), \\ G_{p2}(T_1x_{3n-2},T_2x_{3n-1}T_3x_{3n}) \end{Bmatrix} \\ &\le C\max\begin{Bmatrix} G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p1}(x_{3n+1},x_{3n},x_{3n-1}), \\ G_{p2}(y_{3n-2},y_{3n-1},y_{3n}) \end{Bmatrix} \end{aligned}$$

$$G_{p1}(x_{3n+1},x_{3n},x_{3n-1}) \le C\max\begin{Bmatrix} G_{p1}(x_{3n-2},x_{3n-1},x_{3n}), \\ G_{p2}(y_{3n-2},y_{3n-1},y_{3n}) \end{Bmatrix}$$

and

$$\begin{aligned} G_{p2}(y_{3n+1},y_{3n},y_{3n-2}) &= G_{p2}(T_1S_3y_{3n},T_3S_2y_{3n-1},T_2S_1y_{3n-2}) \\ &= G_{p2}(T_2S_1y_{3n-2},T_3S_2y_{3n-1},T_1S_3y_{3n}) \\ &\le C\max\begin{Bmatrix} G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p2}(y_{3n-1},y_{3n},y_{3n-2}), \\ G_{p2}(y_{3n},y_{3n+1},y_{3n-1}), \\ G_{p1}(S_1y_{3n-2},S_2y_{3n-1}S_3y_{3n}) \end{Bmatrix} \\ &\le C\max\begin{Bmatrix} G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p2}(y_{3n},y_{3n+1},y_{3n-1}), \\ G_{p1}(x,x,x) \end{Bmatrix} \end{aligned}$$

$$G_{p2}(y_{3n+1},y_{3n},y_{3n-2}) \le C\max\begin{Bmatrix} G_{p2}(y_{3n-2},y_{3n-1},y_{3n}), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n}) \end{Bmatrix}$$

From the above inequalities, we obtain

$$G_{p1}(x_{n+1},x_n,x_{n-1}) \le C^n \max\{G_{p1}(x,x_1,x_2),G_{p2}(y_1,y_2,y_3)\}$$
$$G_{p2}(y_{n+1},y_n,y_{n-1}) \le C^n \max\{G_{p1}(x,x_1,x_2),G_{p2}(y_1,y_2,y_3)\}$$
$$P = \max\{G_{p1}(x,x_1,x_2),G_{p2}(y_1,y_2,y_3)\}$$
$$G_{p1}(x_{n+1},x_n,x_{n-1}) \le C^n P$$
$$G_{p2}(y_{n+1},y_n,y_{n-1}) \le C^n P \text{for} n = 1,2,3,\ldots$$

Now, we shall show that $\{x_n\}$ is a G_p Cauchy sequence in X. For each l,m,n, with $l \ge m \ge n$ and $l,m,n \in \mathbb{N}$, we get

$$\begin{aligned} G_{p1}(x_n,x_m,x_l) &\le G_{p1}(x_n,x_{n+1},x_{n+1}) + G_{p1}(x_{n+1},x_{n+2},x_{n+2}) \\ &+G_{p1}(x_{n+2},x_{n+3},x_{n+3}) + \cdots + G_{p1}(x_{m-1},x_m,x_m) \\ &+G_{p1}(x_{l-1},x_l,x_l) - \{G_{p1}(x_{n+1},x_{n+1},x_{n+1}) \\ &+G_{p1}(x_{n+2},x_{n+2},x_{n+2}) + \cdots \\ &+G_{p1}(x_{l-1},x_{l-1},x_{l-1})\} \\ &\le G_{p1}(x_n,x_{n+1},x_{n+1}) + G_{p1}(x_{n+1},x_{n+2},x_{n+2}) \\ &+G_{p1}(x_{n+2},x_{n+3},x_{n+3}) + \cdots + G_{p1}(x_{m-1},x_m,x_m) \\ &+G_{p1}(x_{l-1},x_l,x_l) \\ &\le C^n P + C^{n+1}P + \cdots + C^{l-1}P \\ &\le C^n P(1 + C + \cdots + C^{l-1-n}) \\ &\le \frac{C^n P}{1-C} \end{aligned}$$

By taking the limit as $l,m,n \to \infty$ to both side of the above inequality and from the hypothesis $C < 1$, we have

$$\lim_{l,m,n\to\infty} G_{p1}(x_n,x_m,x_l) = 0$$

It follows that $\{x_n\}$ is a G_p Cauchy sequence in X and by G_p completeness of X, there exists $z \in X$ such that $\{x_n\}$ converges to $z$as $n \to \infty$. i.e.,

$$\lim_{n\to\infty} G_{p1}(z,x_n,x_n) = 0$$

Similarly, $\{y_n\}$ is a G_p Cauchy sequence in Y and by G_p completeness of Y, thereexists $w \in Y$ such that $\{y_n\}$ converges to w as $n \to \infty$. i.e.,

$$\lim_{n\to\infty} G_{p1}(w,y_n,y_n) = 0$$

Now,

$$\begin{aligned} G_{p1}(S_2T_2x_{3n-1},z,z) &\le G_{p1}(s_2T_{2x_{3n-1},x_{3n-1},x_{3n-1}}) + G_{p1}(x_{3n-1},z,z) \\ &-G_{p1}(x_{3n-1},x_{3n-1},x_{3n-1}) \\ &\le G_{p1}(xxx + G_{p1}(x3n-1,z,z) \\ &\le G_{p1}(S_2T_2x_{3n-1}S_1T_1x_{3n-2},S_1T_1x_{3n-2}) \\ &+G_{p1}(x_{3n-1},z,z) \\ &\le C\max\left\{\begin{matrix} G_{p1}(x,x,x),G_{p1}(x,x,x), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n-2}), \\ G_{p1}(x_{3n-1},x_{3n},x_{3n-2}), \\ G_{p2}(T_2x_{3n-1},T_1x_{3n-2},T_1x_{3n-2}) \end{matrix}\right\} \\ &+G_{p1}(x_{3n-1},z,z) \\ &\le C\max\left\{\begin{matrix} G_{p1}(x_{3n-2},x_{3n-2},x_{3n-1}), \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n-1}) \\ G_{p1}(x_{3n-2},x_{3n-1},x_{3n-2}), \\ G_{p1}(x_{3n-1},x_{3n},x_{3n-2}), \\ G_{p2}(y_{3n-1},y_{3n-2},y_{3n-2}) \end{matrix}\right\} \\ &+G_{p1}(x_{3n-1},z,z) \\ &\to 0 \; as \; n \to \infty. \end{aligned}$$

$$\lim_{n\to\infty} S_2\,T_2x_{3n-1} = \{z\} = \lim_{n\to\infty} S_2\,y_{3n-1}$$
$$\lim_{n\to\infty} S_1\,T_{1X3n-2} = \{z\} = \lim_{n\to\infty} S_1\,y_{3n-2}$$
$$\lim_{n\to\infty} S_3\,T_3x_{3n} = \{z\} = \lim_{n\to\infty} S_3\,y_{3n}$$
$$\lim_{n\to\infty} T_2\,S_1y_{3n-2} = \{w\} = \lim_{n\to\infty} T_2\,x_{3n-1}$$
$$\lim_{n\to\infty} T_3\,S_2y_{3n-1} = \{w\} = \lim_{n\to\infty} T_3\,x_{3n}$$
$$\lim_{n\to\infty} T_1\,S_3y_{3n} = \{w\} = \lim_{n\to\infty} T_1\,x_{3n+1}$$

If T_1 is continuous, $\lim_{n\to\infty} T_1\,x_{3n+1} = T_1z = \{w\}$

$$\begin{aligned} &G_{p1}(S_1T_1z,S_2T_2x_{3n-1}S_3T_3x_{3n}) \\ &\le C\max\left\{\begin{matrix} G_{p1}(z,x_{3n-1},x_{3n}), \\ G_{p1}(z,S_1T_1z,x_{3n}), \\ G_{p1}(x_{3n-1},x_{3n},z), \\ G_{p1}(x_{3n},x_{3n+1},x_{3n-1}), \\ G_{p2}(T_1z,T_{2X3n-1},T_{3^x3n}) \end{matrix}\right\} \end{aligned}$$

As $n \to \infty$, we obtain

$$G_{p1}(S_1T_1z,z,z) \le C G_{p1}(z,S_1T_1z,x_{3n})$$
$$G_{p1}(S_1T_1z,z,z) = 0$$
$$S_1T_1z = \{z\} = S_1w$$

Similarly,

$$S_2T_2z = \{z\} = S_2w$$
$$S_3T_3z = \{z\} = S_3w$$
$$T_2S_1w = \{w\} = T_2z$$
$$T_3S_2w = \{w\} = T_3z$$
$$T_1S_3w = \{w\} = T_1z$$

From the above inequalities, we obtain

$$S_1T_1z = S_2T_2z = S_3T_3z = \{z\} = S_1w = S_2w = S_3w$$
$$T_2S_1w = T_3S_2w = T_1S_3w = \{w\} = T_2z = T_3z = T_1z.$$

Hence, $z \in X$ is the common fixed point of $T_1S_1, T_2S_2$and TS. Similarly, $w \in Y$ is the common fixed point of T_2S_1, T_3S_2 and TS.

To prove uniqueness suppose that S_1T_1, S_2T_2 and S_3T_3 have a common fixedpoint z', such that $S_1T_1z' = z', S_2T_2z' = z'$ and $S_3T_3z' = z'$.

Using the inequalities (1) and (2), we have

$$\begin{aligned} G_{p1}(S_1T_1z',S_2T_2z',S_3T_3z') &\le C\max\left\{\begin{matrix} G_{p1}(z',z',z') \\ G_{p1}(z',S_1T_1z',z'), \\ G_{p1}(z',S_2T_2z',z'), \\ G_{p2}(T_1z',T_2z',T_3z') \end{matrix}\right\} \\ &\le C G_{p2}(T_1z',T_2z',T_3z') \\ &\le C G_{p2}(T_2S_1T_1z',T_3S_2T_2z',T_1S_3T_3z') \end{aligned}$$

$$G_{p1}(S_1T_1z',S_2T_2z',S_3T_3z') \le C^2\max\left\{\begin{matrix} G_{p2}(T_1z',T_2z',T_3z'), \\ G_{p2}(T_2z',T_2S_1T_1z',T_3z'), \\ G_{p2}(T_3z',T_3S_2T_2z',T_1z'), \\ G_{p2}(T_3z',T_1S_3T_3z',T_2z'), \\ G_{p1}(S_1T_1z',S_2T_2z',S_3T_3z') \end{matrix}\right\}$$
$$\le C^2 G_{p1}(S_1T_1z',S_2T_2z',S_3T_3z')$$

which yields,

$$G_{p1}(S_1T_1z',S_2T_2z',S_3T_3z') = 0.$$
$$\Rightarrow S_1T_1z' = S_2T_2z' = S_3T_3z' \; as \; C < 1$$
$$\Rightarrow S_1T_1z' = S_2T_2z' = S_3T_3z' = \{z'\}$$

Thus, $T_1z' = T_2z' = T_3z' = \{w'\}$ (say). Then,

$$S_1T_1z' = S_2T_2z' = S_3T_3z' = \{z'\} = S_1w' = S_2w' = S_3w'$$
$$T_2S_1w' = T_3S_2w' = T_1S_3w' = \{w'\} = T_2z' = T_3z' = T_1z'$$

Now,

$$\begin{aligned} G_{p1}(z,z,z') &= G_{p1}(S_1T_1z,S_2T_2z,S_3T_3z') \\ &\le C\max\left\{\begin{matrix} G_{p1}(z,z,z'),G_{p1}(z,S_1T_1z,z') \\ G_{p1}(z,S_2T_2z,z),G_{p1}(z',S_3T_3z',z), \\ G_{p2}(T_1z,T_2z,T_3z') \end{matrix}\right\} \\ &\le C\max\left\{\begin{matrix} G_{p1}(z,z,z'),G_{p1}(z,z,z'), \\ G_{p1}(z,z,z),G_{p1}(z',z',z), \\ G_{p2}(T_1z,T_2z,T_3z') \end{matrix}\right\} \end{aligned}$$

$$\leq C \max \begin{Bmatrix} G_{p1}(z,z,z'), G_{p1}(z,z,z'), \\ G_{p1}(z,z,z), G_{p1}(z,z,z'), \\ G_{p2}(T_1z, T_2z, T_3z') \end{Bmatrix}$$
$$\leq CG_{p2}(T_1z, T_2z, T_3z')$$
$$G_{p1}(z,z,z') = CG_{p2}(T_2S_1w', T_3S_2w, TSw)$$
$$\leq C^2 \max \begin{Bmatrix} G_{p2}(w',w,w), \\ G_{p2}(w', T_2S_1w', w), \\ G_{p2}(w, T_3S_2w, w'), \\ G_{p2}(w, T_1S_3w, w), \\ G_{p1}(S_1w', S_2w, S_3w) \end{Bmatrix}$$
$$\leq C^2 \max \begin{Bmatrix} G_{p2}(w',w,w), G_{p2}(w',w',w), \\ G_{p2}(w,w,w'), G_{p2}(w,w,w), \\ G_{p1}(z',z,z) \end{Bmatrix}$$
$$\leq C^2 \max\{G_{p2}(w',w,w), G_{p1}(z',z,z)\}$$
$$G_{p1}(z,z,z') = C^2 G_{p1}(z,z,z')$$

which yields, $G_{p1}(z,z,z') = 0$. Thus, $z = z'$. Hence z is the unique common fixed point of T_1S_1, T_2S_2 and T_3S_3. Similarly w is the unique common fixed point of T_2S_1, T_3S_2 and T_1S_3. This completes the proof.

Example 2.2. Let $X = [0,1], Y = [1,2]$ and $G_{p1}(x,y,z) = G_{p2}(x,y,z) = \max\{x,y,z\}$. Then (X, G_{p1}) and $G_{p2}(x,y,z)$ are symmetric G_p metric spaces. Define $T_1, T_2, T_3 : X \to B(Y)$ by

$$T_1(x) = x^2 - \frac{1}{3}x + \frac{4}{3}$$
$$T_2(x) = x^2 - \frac{1}{7}x + \frac{8}{7}$$
$$T_3(x) = x^2 - \frac{9}{10}x + \frac{19}{10}$$

Define $S_1, S_2, S_3 : Y \to B(X)$ by

$$S_1(x) = x^2 - \frac{29}{10}x + \frac{14}{5}$$
$$S_2(x) = x^2 - \frac{14}{5}x + \frac{13}{5}$$
$$S_3(x) = x^2 - \frac{23}{10}x + \frac{8}{5}$$

Then,

$$S_1T_1(1) = S_1(1 - \frac{1}{3} + \frac{4}{3}) = S_1(2) = 4 - \frac{29}{5} + \frac{14}{5} = 1$$
$$S_2T_2(1) = S_2(1 - \frac{1}{7} + \frac{8}{7}) = S_2(2) = 4 - \frac{28}{5} + \frac{13}{5} = 1$$
$$S_3T_3(1) = S_3(1 - \frac{9}{10} + \frac{19}{10}) = S_3(2) = 4 - \frac{23}{5} + \frac{8}{5} = 1$$

Therefore, we have

$$S_1T_1(1) = S_2T_2(1) = S_3T_3(1) = 1 = S_1(2) = S_2(2) = S(2)$$

Also

$$T_2S_1(2) = T_2(1) = 1 - \frac{1}{7} + \frac{8}{7} = 2$$
$$T_3S_2(2) = T_3(1) = 1 - \frac{9}{10} + \frac{19}{10} = 2$$
$$T_1S_3(2) = T_1(1) = 1 - \frac{1}{3} + \frac{4}{3} = 2$$

From the above equations, we have

$$T_2S_1(2) = T_3S_2(2) = T_1S_3(2) = 2 = T_1(1) = T_1(2) = T(2)$$

Hence, 1 is a unique common fixed point of T_1S_1, T_2S_2 and T_3S_3 and 2 is a unique common fixed point of T_2S_1, T_3S_2 and TS.

Graphical View:

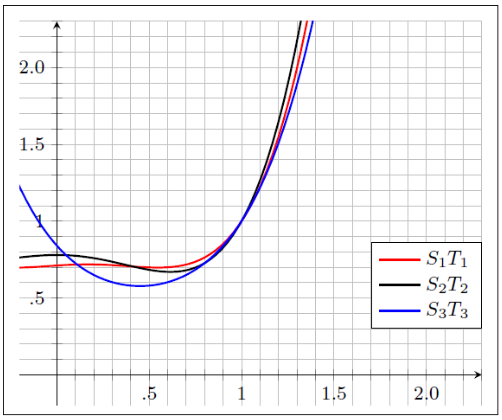

Figure 1: $S_1T_1(1) = S_2T_2(1) = S_3T_3(1) = 1$

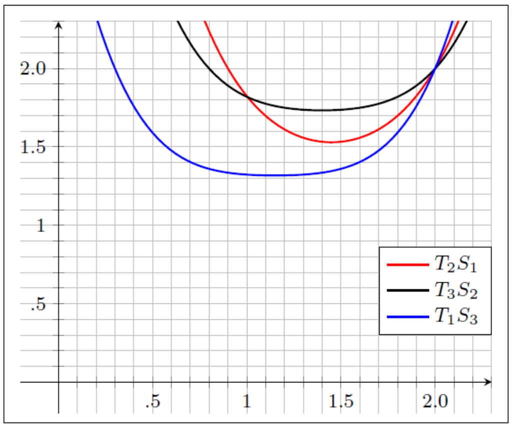

Figure 2: $T_2S_1(2) = T_3S_2(2) = T_1S_3(2) = 2$

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